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A Survey of the Principle and Development of Geometric Function Theory

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Abstract

This article serves as an introduction to the theory of geometric functions. Foundational methodologies and certain advancements within the domain are elucidated with the perspective that the primary audience comprises budding scholars eager to grasp fundamental principles. It commences with rudimentary terminologies and principles, followed by an exploration of selected topics within the realm of univalent functions theory. Various fundamental subsets within the umbrella of univalent functions are outlined. Particular emphasis is placed on the significant category of Caratheodory functions and their interrelations with diverse function classes, particularly the methodologies for deriving conclusions in those alternate classes vis-à-vis the underlying Caratheodory functions. Given the intended audience's novice status, intricate proofs are omitted. Instead, elementary demonstrations are articulated using the most straightforward language possible. Footnotes are incorporated to expound upon points that may not be immediately apparent. References primarily consist of canonical texts. Interested parties are encouraged to consult experts for the latest references, supplementing those cited within the mentioned texts. It is hoped that this exposition will prove beneficial to even seasoned researchers venturing into this field. We commence with the fundamental definition and present a few straightforward examples from the realm of univalent functions. Following cursory

examination of the existing literature, we overview the advancements achieved in addressing specific challenges within this domain.

Keywords: geometric theory, Analytic function, Univalent function.

Introduction

We are interested in power series of the form

$$w = f(\varsigma) = \sum_{j=0}^{\infty} \mathrm{d_j}\varsigma^j \qquad (1)$$

in the complex variable ς =*x*+*iy*, which converge within the unit disk.

If | ς |<1, such a sequence series yields a mapping of Ȩ over a portion of domain *D*. Two questions arise: (Ą) considering the string of coefficients $\mathrm{d_o}, \mathrm{d_1}, \mathrm{d_2}, ..$ What geometric characteristics could we infer concerning *D*? And (B) granted a geometric property of *D*, what conclusions can we draw about the sequence $\mathrm{d_o}, \mathrm{d_1}, \mathrm{d_2}$ ...?

Definition 1. A function *f*(ς) that is orderly in Ȩ is considered univalent in Ȩ if it does not take any duplicate value within Ȩ. Additionally, this function referred to as sįmplė or schlicht? in Ȩ. When *f*(ς) is univalent in Ȩ, we describe the domain *Df*(Ȩ) as univalent domain.

Expressed algebraically *f*(ς) is univalent in Ȩ if the equation w_O=*f*(ς) has no more than one solution in *E* for individually complex w_O.If *f*(ς) is univalent in *E*, then $f'(\varsigma)$≠0in Ȩ. However, one ought to training session caution due to the reverse is j obvious examples, we cite that $f(\varsigma) = -\varsigma$ is univalent in Ȩ, whilst $f_2(\varsigma) = \varsigma^2$ not univạlent in Ȩ. The function $\varsigma + \varsigma^j/j$ is univalent in Ȩ for every favorable integer *j*.The function sinς is univalent in sizeable disk| ς |<*π/2*.not necessarily true.

Problem: The task is to identify a set of conditions on the sequence $\{d_j\}$ that are both necessary and adequate for *f*(ς) be univalent in Ȩ. This open issue is exceedingly challenging, yet partial result have been achieved: some of which will be outlined here.

We note that if *f*(ς) is univalent, then *f*(ς)- d_o, is also univalent, thus we can suppose d_o =0. In (1) without loss of geometrically. That implies translating the domain *D* such that ς =0 maps to $w_O = 0$ under the mapping $w=f(\varsigma)$. Additionally, we observe that $f'(0)=0$, divide byd_1, and then write $f(\varsigma)$ in the form

$$.f(\varsigma) = \varsigma + \sum_{j=2}^{\infty} \mathrm{d}_j \varsigma^j \text{ , } \mathrm{a}_j = \frac{\mathrm{d_j}}{\mathrm{d}} \qquad (2)$$

Geometrically, that entails either reducing or enlarging the domain *D*, and potentially twirling *D*. However, this adjustment don't affect the univalence of function.

During $f(\varsigma)$ takes the pattern as in Equation (2), we refer to the function as normalized. Normalization are feasible, but the series of condition's $F(0)=0$ and $F'(0)=i$ is the most and the one we will adopt here.

Currently, we provide highly significant an instance of normalized univalent function. We initiate our examination accompanied by function

$$g(\varsigma) = \frac{1+\varsigma}{1-\varsigma} \qquad (3)$$

By utilizing the properties of general linear fractional transformations, it becomes evident that $g(\varsigma)$ is univalent in *E*. g(Ȩ) maps onto the half-plane where Re g>0. Upon squaring $g(\varsigma)$, the resulting function $\hbar(\varsigma) = g^2(\varsigma)$ remains in univalent Ȩ also fully maps *E* to The complex cartesian plane. To standardize Ȩ, we subtract $\hbar(0)=1$ and divide by 4times $\hbar'(0)$except for the slit along the negative real axis from 0to$-\infty$.

$$\mathrm{K}(\varsigma)=\frac{1}{4}\left[\left(\frac{1+\varsigma}{1-\varsigma}\right)^2 - 1\right] = \frac{\varsigma}{(1-\varsigma)^2} \qquad (4)$$

An important concept in complex analysis is the Koebe function, which fully covers the complex plane excluding a segment along the negative real axis from 1/4 to negative infinity. Intuitively, this function serves as the most expansive univalent function since appending any open set to its image compromises its univalence. A brief computation

Starting with $\frac{1}{1-\varsigma} = \sum_{j=0}^{\infty} \varsigma^j$and setting in the expression we obtain the power series for $K(\varsigma)$.

$$K(\varsigma)= \sum_{j=1}^{\infty} j\varsigma^j \qquad (5)$$

This series expansion for the "utmost" univalent function indicates promptly

Conjecture 2: If $f(\varsigma)$ is univalent in Ȩ and exhibits the strength series (2), then

$$\left|a_j\right| \le j, \qquad \text{for } j=2,3\ldots \tag{6}$$

This hypothesis has been under investigation for over six decades and remains an unresolved question, notwithstanding its resolution in numerous particular instances. A comprehensive overview of these findings extends well beyond the confines of this manuscript. However, those keen on delving deeper into this subject can explore the literature offered by Spencer and Scheffre [1], Goluzlne [2], Jenkiens [3], Hayman [4], Pommrenke [5], and Schobir [6].

The most significant outcome currently acknowledged is credited to D. Horowitz [7], who demonstrated that

$$\left|a_j\right| \le 1.065\, j \tag{7}$$

Employing an exceedingly profound approach attributed to Carl FitzGerald [8].

The inquiry posed by conjecture1 gives rise to a plethora of associated inquiries, with some remaining unresolved while others have been thoroughly addressed. The Koebe function stands as the "epicenter of the domain" due to its recurrent appearance; hence, a theorem in this realm garners considerable intrigue if it eschews the utilization of the Koebe function or we shall delve into a selection of these theorems and conjectures, exploring their intricacies and potential implications.

The unit disk: We consider the domain of g to be the unit circle Ę $= \{\varsigma : \lceil \varsigma \rceil < 1\}$. Why this choice? Absolutely. The Riemann Mapping Theorem ensures of that kind region in the complex plane might be conformably represented to any other analogous characteristics. In simpler terms, Riemann proved the existence of an analytic function capable of transforming one simply connected area to another with analogous properties. Initially, Riemann's groundbreaking assertion seemed somewhat lacking in significance or robustness till the emergence the theory of univalent functions saw a significant change in 1907 when Koebe discovered that analytic and univalent mapping possess the desirable attribute described in Riemann's assertion [9]. If $\varsigma_o \in D$,

Then, a distinct analytic and univalent g exists, which maps the open unit disk D onto the regionĘ, ensuring that $g(\varsigma_0)=0$ and $g'(\varsigma_o)>0$

Hence, due to the univalence and consequent conformity of g, complexities regarding the geometry within any simply connected domain in the complex plane need not be a concern. Various issues concerning such domains can ultimately be simplified to the particular case of the open unit disk.

Normalization 3 The function g is adjusted in normalized manner to satisfy the condition that

(I) It maps the origin to zero, meaning $g(0)$ equals zero,

(II) Its derivative evaluates to 1 at the origin, denoted as $g'(0) = 1$. This is evident based on Riemann's assertion which suggests that, devoid of any compromise in generality, we can set $\varsigma_o = 0$. Thus, the assertion simplifies to:

In this cases where D encompasses, then a singular function g exists, both analyticity and univalence, mapping D upon the open unit disk Ȩ. This function is unique, and it holds the properties $g(0) = 0$ and $g'(0) > 0$.

Achieving the requirement $g(0) = 0 \, and \, g'(0) > 0$ is precisely the purpose of normalization. To accomplish this, let's define the function g as follows:

$$f(\varsigma) = \frac{g(\varsigma) - \mathrm{d}_o}{\mathrm{d}_1}$$

The condition $\mathrm{d}_1 \neq 0$is not true for all analytic functions g. For instance, the analytic function $g(x) = \varsigma^2$serves as a counterexample. However, there are numerous other functions that can be normalized. It is evident that class of normalizable analytic functions is not empty.

Thankfully, there is a subset of these functions that possess a desirable property. These are the functions that are injective or univalent. In the language of geometric functions, such functions are referred to by different names, including univalent, simple schliecht, or odnolistne. They are functions that don't assign the equal value two times. In other words, if ς_1 and $\varsigma_2$are points in the domain D of $g, g(\varsigma_2)$ and $g(\varsigma_1)$ are distinct whenever $\varsigma_1 \neq \varsigma_2$

ndeed, it's not overly complex to visually ascertain that f is injective if and only if$\acute{f}(\varsigma) \neq 0$, meaning it lacks a zero d_1This implies that f never makes a turn within its domain.

An elementary analytic proof of this assertion follows from the assumption that if there were such turns, then for sufficiently small ς, g could be approximated (neglecting terms of(ς^3) as negligible) by:

$$g(\varsigma) \approx \mathrm{d}_o + \mathrm{d}_2\varsigma^2$$

in which case g loses univalence

Now, with the assurance of g's univalence, the desired normalization can be achieved. Let's denote the normalized functions as *S*. Furthermore, we represent them as:

$$f(\varsigma) = \varsigma + \mathrm{a}_2\varsigma^2 +, \dots \qquad (8)$$

Where $\mathrm{a_j} = \frac{\mathrm{d_j}}{\mathrm{d_1}}$, j=2,3…and $\mathrm{d}_1 = 0$

The range of f 4

Do we differ on whether "geometric function theory" accurately describes this field of study? No, we don't .As Macgregor elucidates, the term accurately captures the essence of this discipline. The importance of geometric concepts and quandaries "geometric function theory" accurately characterizes the essence of the field within complex analysis. While similar thoughts exist in real analysis, geometry exerts a profound influence on complex analysis, rendering it an indispensable and fundamental aspect according to Duren [10].

The synergy between geometric concepts and analytical methods is a captivating feature of complex function theory. The examination of univalent functions further elucidates these intricate connections between mathematical structures and spatial properties.

The domains of these functions illustrate a variety of graceful geometries and traditional classifications. For instance, if f is a normalized analytic and univalent function in E, then its domain encompasses a disk |w|<δ. additionally, some of these functions represent shapes that are star-shaped, close-to-star-shaped, convex, close-to-convex, or linearly accessible. Certain functions manifest these shapes in particular directions, while others uniformly, and some concerning conjugate symmetric points, among other variations. These functions, whose domains delineate specific geometries, are thus termed geometric functions. Moreover, the examination of these functions falls under the purview of geometric unction theory.

In particular, a region of the complex plane demonstrates star geometry concerning a specified point within it if every other point in the region is visible from that fixed point. Put simply, any ray or line segment originating from the fixed point and extending to any other point within the region stays entirely within the region. If a region exhibits star geometry concerning every point within it, it is termed convex. Essentially, this implies that the line segment connecting any two points within the region remains wholly contained within the region itself..

Assignments whose ranges exhibit luminary geometry are termed star functions, while those whose ranges display convex geometry are referred to as convex functions. This concept holds true across various classes of functions.

Some subjects of inquiry

The theory of geometric functions serves as a powerful tool for solving a broad spectrum of problems in mathematical analysis. Its results find application across numerous in the realms of mathematics, the physical sciences, and engineering, it's crucial to consider the seminal compilation by S. Berenardi:

Bibliography of Schliecht functions:

Courant Institute of Mathematical Sciences, New York University, 1966.Part II, Courant Institute of Mathematical Sciences, New York University, 1977.Reprinted with Part III added by Marinir Publishing, Tampa, Florieda, 1983.

These references serve as valuable sources for studying and exploring a wide range of schliecht functions and their applications in the fields of mathematics, physics, and engineering, enumerating the diverse subject areas of geometric function theory, accompanied by a compilation of numerous research outputs in those domains.

We now commence our discussion of these univalent functions by acknowledging the abundance of their existence in nature. To such an extent that the straightforward definition, $f(\varsigma_1) = f(\varsigma_2) \Longrightarrow \varsigma_1 = \varsigma_2$ or its equivalent $\varsigma_1 \neq \varsigma_2 \Longrightarrow f(\varsigma_1) \neq f(\varsigma_2)$, cannot generally be employed to recognize, identify, isolate, many of them. Consequently, this challenge has spurred the development of several new methods in mathematical analysis aimed at achieving this objective. In a special, methods fall under what commonly referred to as.

Sufficient conditions for univalence 1

Result in this direction are as researchers in this field. They persist to be published incessantly, with no f finale in vision. Among these results, one notable and straightforward theorem is the Noshero-Warschewski Theorem: If f is analytic in a domain D and Re$\acute{f}(\varsigma)$>0then f is univalent in that domain.

He evidence supporting the aforementioned univalence conditain hinges on the premise that the function f is established over a line fragment connecting either of the two separate points within domain, designated as $\Omega{:}\mathcal{L}\varsigma_2+(1-\mathcal{L})\varsigma_1$Therefore, through the transformation

$\varsigma=\mathcal{L}\varsigma_2+(1-\mathcal{L})\varsigma_1$

$(d\varsigma=(\varsigma_2-\varsigma_1)d\mathcal{L})$ we have

$f(\varsigma_2)-f(\varsigma_1)=\int_{\varsigma_1}^{\varsigma_2}\acute{f}(\varsigma)d\varsigma=(\varsigma_2-\varsigma_1)\int_0^1\acute{f}(\mathcal{L}\varsigma_2+(1-\mathcal{L})\varsigma_1 d\mathcal{L}\neq 0$

Since Re $\acute{f}(\varsigma)>0$

Indeed, the claim of the Noshero-Warschewski theorem is encapsulated within an equivalent yet broader assertion, as follows:

[Close--convexity [11]] If f is analytic within a domain D, if there exists a convex function g such that Re $\acute{f}$ $(\varsigma)/g'(\varsigma)$ > 0 for all ς in D, then f is univalent within D.

Arguably more than any other, this particular subject has been instrumental in the identification of numerous subfamilies the class of univalent in the unit dick. Several of these subclasses are deliberated upon.

Closely related to this is the inquiry into wherein do metamorphoses maintain univalence within the unit disk. Among the majority fundamental ones are: conjugation, $\overline{f(\bar{\varsigma})}$rotation,$e^{-i\theta}f(e^{i\theta}\varsigma)$; dilation, $f(r\,\varsigma)/r$ for 0<r<1; disk automorphism, $\left[f(\frac{(\varsigma+\sigma)}{(1+\bar{\sigma})-f(\varsigma)}\right]/\left[(1-|\sigma|^2 f(\acute{\sigma})\right]$, σ∈E; omitted-value $\frac{\xi f(\varsigma)}{[\xi-f(\varsigma)]}$; $f(\varsigma)\neq\xi,\xi\in$ Ȩ square root, and the configuration transformations, $\phi(f(\varsigma))$, ϕ is still analytically normalized and univalent, while being constrained within the range of f. These transformations can all be readily verified using the given definition $f(\varsigma_1)=f(\varsigma_2)\Rightarrow \varsigma_2 = \varsigma_2$, except for the square root transformation, which necessitates.

Advancements in the field have prompted the exploration of more complex transformations, especially those arising as solutions to certain linear or nonlinear differential equations. Among these, The Lebira integral transform is recognized as the most basic form denoted.

$$J(f) = 2\varsigma \int_0^{\varsigma} f(\mathcal{L})\, d\mathcal{L}, ([9]). \qquad (9)$$

This integral transform, described in equation (2.1), encapsulates a fundamental technique in the evolution of the subject, allowing for deeper insights into the behavior of univalent functions. The Lebira integral emerges as the outcome of the initial linear differential equation of the first order

$$\varsigma \acute{f}(\varsigma) + f(\varsigma) = 2g(\varsigma).$$

Numerous other integrals have been explored, often serving as extensions of the Lebira integral. These transformations encompass nature delve into the characteristics and properties concerning solutions to specific differėntial equạtion′s, particularly concerning scenarios where f possesses certain known properties. These investigations also explore the extent to which such properties can be transferred to the solutions of these equations.

Radius problems 2

If we consider that certain transformations or geometric conditions that fail to maintain univalence, such as those within the unit disk.It naturally leads to the question of whether such transformations or conditions may uphold univalence in a smaller subdisk $Ę_o = \{ \varsigma : |\varsigma| < \rho < 1\} \subset Ę$. Problems of this nature are commonly referred to as "radius problems." Specifically, these problems revolve around determining. The radius, ρ, represents the largest subdisk, $Ę_o$ within which specific transformations of a univalent function f or certain geometric conditions ensure univalence. This radius, ρ, is commonly referred to as the "radius of univalence"among other designations. The notion of the radius of univalence extends beyond the realm of univalence alone, giving rise to numerous related questions such as the radius of star likeness, and close--convexity.

A fundamental result in this line of inquiry is:

[Noshiro,Yameguchi [12]] If f satisfies $Ref(\varsigma)/\varsigma > 0$ in E, then its univalent in the subdisk $|\varsigma| < \sqrt{2} - 1$

Convolution or hadamard product [13, 14] 3

Let's define two analytic functions within the unit circle, denoted as $f(\varsigma) = a_o + a_1\varsigma + a_2\varsigma^2 \ldots$ and $g(\varsigma) = d_o + d_1\varsigma + d_2\varsigma^2 \ldots$The operation of convolution (or Hadamard product) between $f(\varsigma)$ and $g(\varsigma)$, denoted as $(f * g)(\varsigma)$, is given by:

$$(f * g)(\varsigma) = \varsigma + \sum_{j=2}^{\infty} a_j d_j \varsigma^j$$

(This concept originates from the integral representation:

$h(r^2 e^{i\theta}) = (f * g)(r^2 e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i(\theta-\mathcal{L})}) g(re^{i\mathcal{L}}) d\mathcal{L},\ r<1$

The convolution operation has demonstrated its utility in addressing various challenges within the realms of analytic and univalent function theory. It has particularly facilitated the closure of function families under specific transformations. This is because numerous transformations of f can be represented as convolutions of f with other analytic functions, often with predefined behavior. Thus, it is natural to aspire to explore the convolutional properties across different function classes. For instance, the Libera transform (2) can be expressed as the convolution $J = g * f$, where $g$an analytic function is.

$$g(\varsigma) = \varsigma + \sum_{j=2}^{\infty} \frac{2}{j+1} \varsigma^j$$

The function g possesses certain appealing geometric properties, which could potentially transfer to the Libera transform through convolution, as extensively documented in the literature. This suggests promising avenues for further exploration and study.

Coefficient inequalities 4

Upon closer examination of the series expansion of f, it becomes evident that various properties such as growth, distortion, and even univalence could be influenced, or indicated, by the magnitudes of its coefficients. Duren articulates:

In its most general from, the coefficient problem encompasses entails determining the region in C^{j-1}occupied by the points $a_2, a_3 \ldots a_j$for all $f \in S$ Deriving such precise analytic information from the geometric hypothesis of univalence proves to be exceedingly challenging.The majority of the content in this section of the article is drawn from the comprehensive survey by Duren [10]. This source provides extensive coverage of the intricate issues surrounding coefficient problems within the field, making it a valuable resource for in-depth exploration.

The coefficient problem has undergone a reformulation, focusing on the specific task of estimating $|a_j|$, perhaps no problem within the field has presented as formidable a challenge to its practitioners as the coefficient problem. As early as 1916, Biebrebach conjectured [15] that the modulus of the jth coefficient of a univalent function is less than or equal to that of the Koebe function:

For every function $f \in S$,$|a_j| \leq j$ for j=2,3…

Except for the Koebe function or its rotations, strict inequality applies for all j Biebrebach himself established that $| a_2 | \leq 2$ as straightforward consequence of the area theorem [10], credited to Greomwall, was resolved in 1923 by Loewner for the third coefficient. Garabedian and Schiffier resolved the fourth coefficient in 1955, with Chirzynski and Schiffier offering an elementary proof in 1960. Although proofs for the fifth and sixth coefficients surfaced later, the conjecture lingered until 1985 when De Branges provided the definitive solution. Astonishingly, the conjecture endured for sixty-nine years prior to resolution. Nevertheless, this extended period saw considerable advancements as it spurred the development of innovative methods and techniques in both the specialized theory and broader complex analysis.

The endeavor for the utmost precision in determining the coefficients of odd univalent functions, often represented through the square root transformation of a function, is deeply interwoven with the Bieberbach conjecture$f \in S$.

$$l(\varsigma) = \sqrt{f(\varsigma^2)} = \varsigma + c_3\varsigma^3 + c_5\varsigma^5 \ldots$$

For odd univalent functions, Littlwood and Paley demonstrated in 1932 that for each j, the modulus $|c_j|$ is bounded above by an absolute constant A. Their method demonstrated that this constant A and they appended a footnote suggesting, undoubtedly, the true bound is provided by A=1, which later became apparent as the Littlwood-Paley conjecture. The validity of this conjecture for definite subclasses of S masked its falsehood broadly until as promptly as 1933, approximately a year after its proposition, when it was refuted by what became known as the Feket-Szegö problem.

Coefficient related problems 5

These tasks encompass identifying the sequential coefficient relationships and delineating the range of coefficient variability.

Growth, distortion and covering [16] 6

The notion of growth in an analytic function f pertains to the magnitude of its image domain, denoted as $|f(\varsigma)|$. The term "distortion" emerges from the geometric interpretation of $|f'(\varsigma)|$ as the infinitesimal scaling factor of arclength under the mapping f, or from the Jacobian $|f'(\varsigma)|^2$ as the infinitesimal scaling factor of the area of the image domain. The notion of encompassing by a function f denotes the area of the image domain that it covers. For the comprehensive array of univalent functions, it is established that the domain of every member function encompasses the disk $|\xi|<1/4$. This proposition, initially introduced by Koebe in 1907, is widely known as the Koebe one-Quarter Theorem. It stems from the Bieberbach Theorem concerning the second coefficient of functions in S and their neglected-value transformation.

Partial sums 7

The inquiry into the partial sums $S_j(\varsigma) = \varsigma + a_2\varsigma^2 + \cdots + a_j\varsigma^j$ of the series expansion of f concerns the degree to which established geometric characteristics of f are preserved in its partial sum. Another relevant finding from Yamaguchie is as follows:

[Yamaguchie [13]]: If f satisfies $Re\, f(\varsigma)/\varsigma > 0$ in E, then the jth partial sum $S_j(\varsigma) = \varsigma + a_2\varsigma^2 + \cdots + a_j\varsigma^j$ is (1-1)in subdisk $|\varsigma|<1/4$

Linear sums or combinations 8

It is also intriguing to explore under what conditions does the linear combination maintain its linearity$(1-t)\Phi + \mathcal{L}\Psi$ preserves certain established geometric properties based on Φ and Ψ when Φ and $\Psi$are geometric quantities associated with f.

Some subclasses of S

Continuing from the preceding discussion regarding some subclasses within the class of univalent functions, it's worth mentioning that the primary rationale for exploring new subclasses stems from the potential to associate certain classes of functions with unique properties not typically associated with other classes. Consequently, numerous areas of investigation are being revisited across various classes of functions to refine, enhance, or extend many established results, particularly with a focus on developing new subclasses. Handful of the notable subclasses of include:

Functions of bounded ($f'(\varsigma)$) 1

These functions are characterized by having derivatives having positive real part, i.e,$Re\, f'(\varsigma) > 0$. As previously mentioned, they are entirely univalent functions. A multitude of results regarding this category can be discovered in the existing literature.

Starlike functions ($\varsigma f'/f$)[16] 2

These functions are characterized by having a positive real part of the quantity $\varsigma f'/f$positive they constitute are entirely univalent functions and also demonstrate convexity. Moreover they are categorized close-to-convex. Results regarding this class of functions can found distributed across various sources.

Convex functions ($1+\varsigma f''/f'$)[21] 3

These functions are characterized by the positivity of the real part of $1+\varsigma f''/f'$. Positive they are entirely univalent functions and also classified as close-to-convex. Literature concerning this class of functions can be found dispersed across various sources.

Close-to-convex ($f\,'/g'$, g is convex)[16] 4

These functions are characterized by having the real part of the quantity $f\,'/g'$, where g is convex, as real. They are completely univalent functions. Literature on this subject is additionally dispersed across various sources. A notable subclass within this category is the bounded turning function, which is a special type of close-to-convex function with $g(\varsigma) = \varsigma$. Numerous other subclasses stemming from the aforementioned classes of functions have been documented in literature. Additionally, several generalizations have emerged through both derivative and integral operators. These operators encompass well-known ones such as the Salegean derivative, Ruschaweyh derivative, and beyond variations thereof.

Caretheodory, related functions and generalizations

A cursory examination of the series expansions for f and various associated geometric quantities like$\frac{\varsigma f'}{f}$, $1 + \varsigma f''/f$, f/g $f\,'/g'$, and many others, which exhibit the property of having positive real parts, suggests the existence of a series form:

$$\hbar(\varsigma) = 1 + c_1\varsigma + c_2\varsigma^2 + \ldots \tag{10}$$

The from (4.1) satisfies $\hbar$ (0)=1 and $Re\hbar(\varsigma) > 0$(positive real parts), f (normalized f(0)=0 and $f'(0) = 1$) and $\hbar$ (normalized by $\hbar$ (0)=1). It is plausible to propose that the discovery of f preceded that of $\hbar$. If not, the discovery of f would likely have prompted exploration into $\hbar$. Investigating $\hbar$ provides valuable insights into the characteristics of f displaying the described geometries.

The function $\hbar$ is termed the Caretheodory function, named after Caretheodory who not only observed the evident though also devoted considerable effort to characterizing it. $\hbar$ can alternatively another perspective is to consider $\hbar$ as a function that is subordinate to the Möbius function.

$$\Omega_o(\varsigma) = \frac{1+\varsigma}{1-\varsigma}$$

The Möbius function plays a central role in the family of functions akin to $\hbar$, assuming the extremum in the most extremal problem for such functions. By subordination, it is implied that there exists a function of unit bound,($\varpi(\varsigma)$ | $\varpi(\varsigma)$ |< 1, normalized by normalized by ϖ (0)=0) such that $\hbar(\varsigma) = \Omega_o(\varpi(\varsigma))$. Consequently, this offers an alternative depiction for $\hbar$, among various others. Specifically, in the context of ϖ, $\hbar$ assumes the following structure:

$\hbar(\varsigma)=1+h(\varsigma)=1+\frac{\varpi(\varsigma)}{1-\varpi(\varsigma)}\quad \varsigma \in E$

[Schwarz's Lemma ([4])]: If ð(ς) is a function bounded in the complex plane E, then for every $0 < r < 1$, |ð(0)| $<$ 1 and | ð($re^{i\varpi}$)| ≤ r, holds true unless ð(ς) $= e^{i\sigma}\varsigma$ for a particular real numberσ.

The previously mentioned theorem is commonly referred to as Schwarz's lemma. It suggests that if ð (ς) is a function bounded in the complex plane E, then the function u(ς) = ð (ς)/ ς is also bounded, implying |u(ς)| < 1. However, it doesn't necessarily ensure normalization by |u(0)| = 0.

[Cartheodory [17]]: If ð(ς) is a function bounded in the complex plane E, then| ð'(ς)| ≤ 1 - $|ð(\varsigma)|^2$.

Studies have shown that any function h can also possess what is commonly referred to as the Herglotze representation, expressed in integral form as follows: $\hbar(\varsigma) = \int_0^{2\pi}\frac{e^{i\mathcal{L}}+\varsigma}{e^{i\mathcal{L}}-\varsigma}d\mu(\mathcal{L})$where $d\mu(\mathcal{L}) \geq 0$ and $\int d\mu(\mathcal{L})=\mu(2\pi)-\mu(0)=1$.The different portrayals of $\hbar$ hold significant applications, which can be explored through additional study.Moreover, the Caretheodory functions are conserved under several transformations: suppose g and h are Caretheodory, then p defined as follows also belongs to the Caretheodory class:

1. $p(\varsigma) = g(e^{i\mathcal{L}\varsigma})$, where $\mathcal{L}$ is real.
2. $p(\varsigma) = g(\mathcal{L}\varsigma)$, where $\mathcal{L} \in [-1,1]$.
3. $p(\varsigma) = g[1 + \mathcal{L}\varsigma/\bar{\varsigma} + t]/g(\mathcal{L})$, where $|\mathcal{L}|<1$.
4. $p(\varsigma) = g(\varsigma) + i\mathcal{L}/1 + i\mathcal{L}g(\varsigma)$, where $\mathcal{L}$ is real.
5. $p(\varsigma) = [g(\varsigma)]^{\mathcal{L}}$, where $\mathcal{L} \in [-1,1]$.

6. $p(\varsigma) = [g(\varsigma)]^{\mathcal{L}} [\hbar(\varsigma)]^{\tau}$, where $\mathcal{L}$, τ, and $\mathcal{L}+\tau$ are all in the range [0,1].

We will now discuss two fundamental coefficient inequalities for $\hbar$. The initial one is grounded in its Herglotze depiction, whereas the latter hinges on its representation through functions bounded by unity. ð(ς).

[Caretheodory([11])]: If $h(\varsigma) = 1 + c_1\varsigma + c_2\varsigma^2 + \cdots$is Caratheodory function, then $|c_j|\leq 2$ for j=1,2,… The Mo¨ebius function achieves evenness in this inequality.

Further advancements the variations have resulted in diverse generalizations of $\hbar$. Janowseki [18] introduced a redefinition of $\hbar$ in terminology of ð, stating that for Invariant real numbers a and d where a∈(−1,1] and d∈[−1,a) (meaning −1≤d<a≤1), h is defined as:

$\hbar(\varsigma)=\frac{1+a\eth(\varsigma)}{1+d\eth(\varsigma)}$ In this context, the Carathéodory function is defined such that it corresponds to the extreme cases where d=−1 and a=1. Additionally, for different selections of parameters "a" and "d", the function " $\hbar$ " still projects the unit disk onto specific regions within the right half-plane.

Arguably, one of noteworthy contributions the advancement of this crucial area of research hinges on the refinement of iterative processes. For two crucial families of functions, the Caratheodory and Janowseki functions [19] .These iterations are defined as follows:

For the Caratheodory family: $p_j = \frac{\varsigma^j}{\alpha}\int_o^{\varsigma} \mathcal{L}^{\alpha-1} p_{j-1}(\mathcal{L})d\,\mathcal{L}$,$j\geq 1$, with $p_j(\varsigma)$=p(ς).

For the Janowseki family: $p_{\sigma,j}(\varsigma) = \frac{\sigma-(j-1)}{\varsigma^{\sigma-(j-1)}}\int_o^{\varsigma} p^{\sigma-1} p_{\sigma,(j-1)}(\mathcal{L})d\mathcal{L}$, $j\geq 1$, with $p_{\sigma,o}(\varsigma)$=$p(\varsigma)$.

These transformations maintain numerous geometric properties within the set of functions possessing the transformations exhibit a positive real part and are normalized by $\hbar(0)=1$. Remarkably, they maintain positivity of real parts, compactness, convexity, and subordination. An intriguing aspect of these transformations is their ability to facilitate exploration of associated function classes, making them simple, concise, and elegant. They have demonstrated significant utility in effortlessly addressing specific problems within the domains of analytic and univalent function theory, highlighting their extraordinary simplicity.

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Numerous techniques have emerged in the field, yet the most fundamental and accessible for beginners is one rooted in the close connection between Carathéodory functions (along with their subsequent developments) and various classes of functions. Significant results have been attained concerning this category of functions. Consequently, exploring diverse problems of geometric functions through an underlying function h has gained widespread acceptance among researchers in this field as a primary technique. The following section presents several examples and provides insight into the methodology of constructing extremal functions.

And there are many researchers currently working in this field, with numerous studies to their credit including H. M. Srivastava[20], Sabir [21], Wanas and Raadhi [22] , Amini and Al-Omari [23], A. R. Juma [24] and some other researchers [24-35].

We will mention some theories in this field.

Theorem 1: If f ∈S satisfies Re ς $f(\varsigma)$>0, then its coefficients satisfy the inequality: $|a_j|\leq 2$. Equality is achieved by the function $f(\varsigma) = 2(\varsigma + 1)/(1 - \varsigma)$

Theorem 2: If f is starlike (Re$\varsigma f'/f$ >0), then $|a_j|\leq j$. Equality is attained by the Koebe function $f(\varsigma)=\varsigma/(1-\varsigma)^2$.

Theorem 3: If f is convex ($Re[\,[1+\ \varsigma f''(\varsigma)/\,f'(\varsigma)] > 0$), then $|a_j|\leq 1$. Equality is achieved by means of Koebe function $f(\varsigma) = \frac{1}{1-\varsigma}$

Theorem 4: If f is close-to-convex ($Ref'(\varsigma)/\,g'(\varsigma) > 0$), where g is convex), then $|a_j|\leq j$. Equality is attained by Koebe $f(\varsigma)=\varsigma/(1-\varsigma)^2$.

Conclusion

Invitation to geometric function theory presents an accessible and comprehensive overview intricate interplay between geometry and functions. Through clear explanations and illustrative examples, readers are guided through fundamental concepts in complex analysis, differential geometry, and topology, demonstrating their practical application in geometric function theory. Suitable for both novice learners and seasoned experts, this text establishes a strong foundation for understanding complex mathematical ideas. Emphasizing practical applications ensures its relevance across diverse fields, from mathematical physics to computer graphics. Overall, it serves as an invaluable resource for students, researchers, and enthusiasts, offering a gateway to a rich and rewarding area of mathematical inquiry while deepening appreciation for the profound connections between geometry and functions.

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