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Polynomial Approximation of an Inverse Cauchy Problem for Modified Helmholtz Equations

Athraa Falih Hasanl, Duaa Jasim, Ibtihal Thabit Jameel, Baydaa Khaleel Mostafa and Fatima M. Aboud

Department of Mathematics – College of Science – University of Diyala, Iraq

scimathms2118@uodiyala.edu.iq

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Abstract

In this paper, an inverse problem for the modified Helmholtz equation arising in heat conduction in the fin is considered. The goal of this paper is the determination of the temperature at the under-specified boundary (inner boundary of an annular domain) benefiting from the accessible part of the boundary with Cauchy data (boundary temperature and heat flux). This problem is solved numerically using the meshless method proposed in [2]. The stability is confirmed by applying a noise for the Cauchy data.

Keywords: Inverse Cauchy problem, Modified Helmholtz equation, Polynomial expansion, Conjugate Gradient method (CGM), Conjugate Gradient Least Square Method (CGLS).

تقريب متعدد الحدود لمسالة كوشي المعكوسة لمعادلات هلمهولتز المعدلة

عذراء فالح، دعاء جاسم، ابتهال ثابت، بيداء خليل مصطفى وفاطمة محمد عبود

قسم الرياضيات – كلية العلوم – جامعة ديالى

الخلاصة

في هذا البحث، تم النظر في مسألة عكسية لمعادلة هلمهولتز المعادلة الناشئة في توصيل الحرارة في الزعنفة الهدف من هذا البحث تحديد درجة الحرارة على الجزء الداخلي من الحدود ذو الشروط الحدودية الغير معلومة (الحدود الداخلية للمجال

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الحلقي) بالاستفادة من الجزء الذي يمكن الوصول اليه من الحدود الممتلكة للبيانات الكوشية (درجة الحرارة الحدودية والتدفق الحراري) يتم حل هذه المشكلة عدديا باستخدام الطريقة اللاشبكية المقترحة في [2] كما تم تأكيد استقرارية الحل التقريبي باضافة ضوضاء على البيانات الكوشية.

الكلمات المفتاحية: مسألة كوشي العكسية، معادلة هلمهولتز المعدلة، مفكوك متعدد الحدود، طريقة التدرج المترافق (CGM)، طريقة التدرج المترافق للتربيعات الصغرى (CGLS)

Introduction

One of the important applications in inverse problem design and optimization is to identify an unknown obstacle and its resistive characteristics. This motivates us to find the temperature on the inner boundary of an annular domain for an inverse Cauchy problem governing by a modified Helmholtz equation.

In this paper, we consider the inverse problem which consists of determining a temperature u on the inner boundary of an annular domain from a given Cauchy data on the outer boundary (boundary temperature and heat flux), assuming that the steady state temperature u satisfies the modified Helmholtz equation governing the heat conduction in a fin

$$\boldsymbol{\nabla^2 u - k^2 u = 0}, \Omega/\boldsymbol{D}$$

from the knowledge of Dirichlet temperature data u and Neumann heat flux data $\frac{\partial u}{\partial n}$ on outer part of the boundary $\partial\Omega$ of Ω, where n is outward unit normal at $\partial\Omega$, and a boundary condition (Dirichlet, Neumann or Robin) on the boundary ∂D of D [Lesnic & Bin-Mohsin, 2004]. This kind of problems is an ill-posed problem. In fact, a problem is well-posed in the sense of Hadamard, in case that the solution exists, unique and stable, [Hadamard, 1923], otherwise if the solution does not satisfy one of these conditions, then the problem is ill-posed, and hence an inverse problem must be formulated to solve this ill-posed problem. In general, the inverse problem is known to be difficult to solve rather than the direct problems.

Moreover, the inverse problems are unstable, [Hadamard, 1923], i.e., in the sense that a small error in measurement of the input data, this can produce a big error in the solution. Recently, inverse problems have been considered in several domains of science, (see [Kubo, 1988]). The Cauchy problem is one of the examples of inverse problems [Chakib et. al., 2018], [Hernandez-Montero et al., 2019], [Isakov, 2017], [Kabanikhin, 2012], [Lavrent'ev, 1986], [Liu & Wang, 2018], [Nachaoui et. al., 2021]. For this type of problems, the boundary conditions (Dirichlet, Neumann) are known only on some part of the boundary (accessible part), and on the other part of the boundary there is no given data so this part is said to be under-specified or un-accessible.

For these complications, a suitable algorithm must be chosen to be able to reduce the ill-posedness of this type of problem. In the last two decades, several methods have been developed for solving the Cauchy problem of the Helmholtz equation. Here we recall some of these methods, the truncation method [Yang, 2019], the conjugate gradient method [Marin et al., 2003b], the meshless generalized finite difference method [Hua et. Al., 2017], the Landweber method [Yang et. al., 2017], the fractional Tikhonov regularization method [Qia & Feng, 2017].

In fact, the dependence of the numerical solutions of direct Helmholtz equation on the physical parameters k has a remarkable effect on the quality of approximation. For more details about this see [Ihlenburg & Babuška, 1995 and 1997]. Some methods have been proposed to solve Cauchy Helmholtz equation for some big parameter k, see [Berntsson et. Al, 2014, 2017, 2018], [Karimi & Rezaee, 2017] and [Qian & Feng, 2017]. An alternating algorithm based on relaxation of alternating algorithms has proposed by [Jourhmane & Nachaoui, 1996]. An effective relaxed alternating procedure proved the convergence for all values of wave number k in the case of Helmholtz equation and accelerate the convergence in the case of modified Helmholtz equation in [Berdawood et. Al., 2021], the authors proved that for any value wave number k we find an interval of relaxation parameter in which the convergence is assured.

The aim of this paper is to explore a method based on polynomial expansion for the approximation of the solution of a Cauchy problem for a modified Helmholtz-type equation in a bounded domain enclosed by a smooth boundary. In this work, the temperature of the un-accessible inner boundary is approached using a meshless method proposed in Rasheed et al. [Rasheed et al. 2021]. This method was proposed in [Rasheed et. al, 2021] to solve an inverse Cauchy problem and by [Jameel et al., 2022] to solve a Cauchy problem Helmholtz equation.

The next sections of this paper are organized as follows; in section 2 we recall the inverse Cauchy problem for modified Helmholtz equation. Our proposed approximate method is given in section 3. Some numerical methods is considered and illustrated by applying them, for some examples in section 4.

2. Inverse Cauchy problem for the modified Helmholtz equation

Let us consider the domain $\Omega \backslash D \subset R^2$ with

$\Omega = \{(r,\theta) : 0 \leq r < 1, 0 \leq \theta \leq 2\pi\}$

$D = \{(r,\theta) : 0 \leq r < \beta, 0 < \beta < 1, 0 \leq \theta \leq 2\pi\}$

Let us consider $\Omega \subset R^2$ with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$

$\Gamma_1 = \{(r,\theta) : r = \rho_e(\theta)\ 0 \leq \theta \leq 2\pi\}$ outer boundary

$\Gamma_2 = \{(r,\theta) : r = \rho_i(\theta)\ 0 \leq \theta \leq 2\pi\}$ inner bounda

We consider the inverse Cauchy problem for the modified Helmholtz equation giving in the following

$$\Delta \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) - \boldsymbol{k}^2 \boldsymbol{u}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{0}\ \Omega \backslash \boldsymbol{D} \quad \textbf{(1)}$$

$$\boldsymbol{u}(\boldsymbol{\rho},\boldsymbol{\theta}) = \boldsymbol{h}(\boldsymbol{\theta})\ \boldsymbol{on}\ \partial\Omega = \Gamma\mathbf{1} \quad \textbf{(2)}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(\boldsymbol{\rho},\boldsymbol{\theta}) = \boldsymbol{g}(\boldsymbol{\theta})\ \boldsymbol{on}\ \partial\Omega = \Gamma\mathbf{1} \quad \textbf{(3)}$$

Note Cauchy data $u(x,y)$ and $\frac{\partial u}{\partial n}(x,y)$ are given on the accessible part of the boundary of the domain **Ω**. Where $h(\theta), g(\theta)$are given function.

Note that the part $\Gamma 1$ is (over determined) two boundary conditions are specified, while $\Gamma 2$ is under determined (no boundary condition is specified). The inverse problem for the modified Helmholtz equation is formulated to determine the temperature u on the interior under-determined boundary $\Gamma 2$. Recalling that the normal derivative of u, denoting by $\frac{\partial u}{\partial n}$, can be expressed in the following from (Liu&Kuo,2016), Rasheed et al. (2021):

$$\partial_n u(\rho,\theta) = \eta(\theta)\left[\frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}\right] \quad \text{..........} (4)$$

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta)+[\acute{\rho}(\theta)]^2}} \quad \text{..........} (5)$$

The normal derivative $\partial_n u(x,y)$ we can also express in terms of $\partial_x u$ and $\partial_y u$ by (Liu&Kuo,2016), Rasheed et al. (2021):

$$\partial_n u = \eta(\theta)\left[\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right]\partial_x u + \eta(\theta)\left[\sin(\theta) - \frac{\rho'}{\rho^2}\cos(\theta)\right]\partial_y u \quad \text{..........} (6)$$

3. Approximation of the solution by polynomial expansion

The solution $u(x,y)$ can be expressed as a polynomial expansion

$$u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{i,j} x^{i-j} y^{j-1} \quad \text{..........} (7)$$

To find $u(x,y)$, the coefficients c_{ij} must be determined. The number of these coefficients is n= $\frac{m(m-1)}{2}$, note that the maximal order of the above polynomial is $m-1$.

Using equation (7) we find $\partial_x u$, $\partial_y u$ and Δu

$$\partial_x u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} cij\,(i-j)\,x^{i-j-1}y^{j-1} \quad \text{..........} (8)$$

$$\partial_y u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} cij\,(j-1)\,x^{i-j}y^{j-2} \quad \text{..........} (9)$$

$$\Delta u(x,y) - k^2 u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} cij[(i-j)(i-j-1)\,x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} \quad \text{..........} (10)$$

Firstly, the coefficients c_{ij} in (8) can be expressed as a n-dimensional vector c with the components $c_k, k = 1,\ldots,n$. In fact, the coefficients c_{ij} are reordered taking in consideration that $i = 1,\ldots,m$ and $j = 1,\ldots,i$, each index ij corresponding one index k by taking $k = \frac{i(i-1)}{2} + j$. The term $u(x,y)$ can be expressed as an inner product of the vector a^T with c, that

$$\boldsymbol{u(x,y) = [1\ x\ y\ x^2\ xy\ y^2\ x^3 x^2 y\ xy^2\ y^3\ \ldots]} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{a^T c} \quad \ldots\ldots (11)$$

We replace (8) and (9) in to (6) gives us an expression of $\partial_n u(x,y)$. Similarly, for each point on the accessible part of the boundary $\Gamma_1$the normal derivative $\partial_n u(x,y)$ can be expressed as an inner product of a vector e with c, such that the $l-th$ component of e is given by:

$$\boldsymbol{e_l = \eta(\theta)\left[(i-j)x^{i-j-1}y^{j-1}\left(\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right) + (j-1)x^{i-j}y^{j-2}\left(\sin(\theta) - \frac{\rho'}{\rho^2}\cos(\theta)\right)\right]} \quad \ldots\ldots (12)$$

For l =1, …, n and by keeping the same coefficients i, j for those used to calculated e_l from cij. Now for each point in the domain the term $\Delta u(x,y) - k^2 u(x,y)$ can be expressed from (10) as an inner product of a vector d with c, where the $l-th$ component d_l , l =1, …, n is given by:

$$\mathbf{d_k = (i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} - k^2(x^{i-j}y^{j-1})} \quad \ldots\ldots (13)$$

Choosing n₁ points on boundary $\Gamma 1$, say $(xi\,, yi) = (\cos(\theta_i)\,, \sin(\theta_i))$, $i = 1,2,\ldots,n_1$ to verify the boundary condition (2)-(3) and also we take n_2 points in the domain $\Omega\backslash D$, say $(x_j, y_j), j = 1,\ldots,n_2$ to satisfy the equation (1). So, we obtain the linear system

$$\boldsymbol{Ac = b} \quad \ldots\ldots (14)$$

So, the vector b is of longer $(2n_1 + n_2) \times 1$ and A is $(2n_1 + n_2) \times \frac{m(m+1)}{2}$) matrix given respectively by

$$\boldsymbol{A = \left[a_1^T .. a_{n1}^T\ e_1^T .. e_{n1}^T\ d_1^T .. d_{n2}^T\right]\ b = [h(\theta_1)\ldots h(\theta_{n1})\ g(\theta_1)\ldots g(\theta_{n1})\ 0\ 0\ ]} \quad \ldots\ldots (15)$$

4 Solving the linear system

To solve the linear system $Ac = b$, we use the well-known Conjugate Gradient method (CGM) and the Conjugate Gradient least square method (CGLS**) (see [8]).**

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4.1 Algorithms of the Conjugate Gradient method (CGM) and the Conjugate Gradient least square method (CGLS)

The conjugate gradient method will at first be described as an iterative method to solve a linear system of equations. Ax=b, A∈ $R^{n,n}$ symmetric and positive definite

Conjugate Gradient method (CGM)	Conjugate Gradient least square method (CGLS)
Algorithm 1: conjugate Gradient Method (CGM Given a positive definite and symmetric system of equation $Ax = b$, and an initial solution x_0 , let $\beta - 1 = 0, p - 1 = 0, r_0 = b - Ax_0$ and $k = 0$. 1. let $p_k = r_k + \beta_{k-1}p_{k-1}$ 2. let $\alpha_k = \frac{\|r_k\|_2^2}{p_k^T A p_k}$ 3. let $x_{k+1} = x_k + \alpha_k p_k$ 4. let $r_{k+1} = r_k + \alpha_k A p_k$ 5. let $\beta_k = \frac{\|r_{k+1}\|_2^2}{\|r_k\|_2^2}$ 6. let $k = k + 1$ 7. Repeat the previous steps until convergence.	Algorithm 2: Conjugate Gradient least Square Method (CGLS) Given a least squares problem min $\|Gm - d\|_2$,let $k = 0, xo = 0, p - 1 = 0, \beta_{-1} = 0, s_0 = -b, and$ $r_0 = A^T s_0$ 1.let $p_k = -r_k + \beta_{k-1}p_{k-1.}$ 2.let $\alpha_K = \frac{\|r_k\|_2^2}{(p_k^T A^T)(Ap_k)}$ 3.let $x_{k+1} = x_k + \alpha_k p_k$ 4. let $s_{k+1} = s_k + \alpha_k A p_k$ 5. let $r_{k+1} = A^T s_{k+1}$ 6. let $\beta_k = \frac{\|r_{k+1}\|_2^2}{\|r_k\|_2^2}$ 7.let $k = k + 1$ 8. Repeat the pervious steps until convergence.

4.2 Stopping criterion and Initial guess

For any numerical method, it is very important to choose the condition in which the algorithm can stop, so we choose the following stopping criteria:

$$\|r_i\| < Tol \quad \text{……………………………………} \quad (16)$$

$$\frac{\|r_i\|}{\|b\|} < Tol \quad \text{……………………………………} \quad (17)$$

Also, we need to define the initial data for the algorithms CGM and CGLS, for this we give as a guess for u on the under –specified boundary $\Gamma_{2,}$, the zero vector as an initial guess in the first iteration.

5 Numerical results and discussion

In this section we study some examples and solve them numerically to illustrate the ability of our proposed method on the under-specified boundary Γ2. Considering two cases of exact solution (polynomial and non-polynomial), this exact solution is used to calculate the function F, its trace h and its normal derivative g on Γ1 . By using these exact given data on the accessible part of the boundary Γ1 and with zero initial data, and by using the CGM and $CGLS$ like-methods described in section 4.2 with a propitiate tolerance and stopping criteria.

5.1 Polynomial exact solution.

In the following we study a Cauchy problem of modified Helmholtz equation with a polynomial exact solution.

Example 1:

We consider the Cauchy problem for a modified Helmholtz equation with exact solution $u(x,y) = x^2 + y^2$, defined in an annular domain with the constant radius $\rho_{e=}1 \; and \; \rho_i = 0.5 \; and \; \beta = 2$. This problem is over–specified on the outer boundary $\Gamma_1 = \{(x,y): x^2 + y^2 = 1\}$ for which we have the following Cauchy data $h = x^2 + y^2, g = 2x \cos(\theta) + 2y \sin(\theta)$. We study different cases for a different physical parameter $= \sqrt{100}, \sqrt{52}, \sqrt{25.5}, \sqrt{15}$. For the numerical computations, we take $n_1 = 100 \; nr = 20$ and so $n_2 = 2000$ and we take $m = 2, \dots, 10$. We compare the results obtained by using the both algorithms CGM and CGLS with $tol = 10^{-15}$.

The obtained results are presented in tables 1,2,3,4.

Table 1: $k = \sqrt{15}$

m	Error by CGM	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	1.0093e-13	3	1.0509e-15	3
4	9.7151e-14	3	4.5193e-16	3
5	1.5007e-12	11	1.3264e-14	9
6	1.7418e-12	12	1.4619e-14	10
7	1.3013e-11	32	3.9838e-14	23
8	8.9979e-12	35	1.8069e-13	26
9	9.1852e-12	77	2.6307e-13	56
10	1.5066e-11	83	4.4879e-13	70

Table 2: $\boldsymbol{k = \sqrt{25.5}}$

m	*Error by CGM*	*Iteration*	*Error by CGLS*	*Iteration*
2	-	-	-	-
3	*1.0457e-13*	*4*	*4.0898e-16*	*3*
4	*1.0725e-13*	*3*	*4.0883e-16*	*3*
5	*1.3055e-12*	*10*	*4.0993e-15*	*9*
6	*1.3974e-12*	*13*	*1.7169e-15*	*9*
7	*1.2944e-11*	*29*	*1.4054e-13*	*22*
8	*1.1588e-11*	*33*	*3.5495e-14*	*29*
9	*2.4493e-11*	*81*	*5.2835e-13*	*67*
10	*3.0617e-11*	*113*	*9.9042e-13*	*71*

Table 3: $\boldsymbol{k = \sqrt{52}}$

m	Error by CGM	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	3.2029e-14	3	6.5033e-15	3
4	3.4359e-14	3	6.4978e-15	3
5	3.1667e-13	9	5.4102e-15	9
6	3.1551e-13	9	7.6848e-15	9
7	2.8391e-12	30	1.3447e-13	25
8	1.5168e-12	32	9.1517e-14	23
9	1.0776e-10	159	7.7372e-13	66
10	1.3800e-10	128	7.3103e-13	79

Table 4: $k = \sqrt{100}$

m	Error by CGM	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	7.5497e-14	3	9.9582e-15	3
4	7.3731e-14	3	8.5414e-15	3
5	5.6107e-13	9	4.5386e-15	9
6	5.4189e-13	10	1.7544e-15	9
7	1.7268e-12	30	9.6685e-14	21
8	1.8537e-12	29	7.0721e-14	27
9	5.6959e-11	102	2.8958e-13	66
10	5.6061e-11	94	4.2904e-13	71

The influence on the accuracy of the number of external boundary and the internal domain points n_1, n_2 when increases is explored in table 5,6,7,8 where we take $n_1 = 300\ nr = 50$, $n_2 = 15000$,and compare CG and CGLS for different m.

For the results presented in tables 5,6,7,8, we observe that from m=3, for every k ($\sqrt{100}, \sqrt{52}, \sqrt{25.5}, \sqrt{15}$), the CGLS is very accurate when n_1, n_2 is increases

Table 5: $k = \sqrt{15}$

m	Error by CGM	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	4.4495e-13	5	4.8797e-14	3
4	4.4194e-13	5	4.9656e-14	3
5	9.8195e-12	10	2.0279e-14	9
6	1.0516e-11	11	4.4876e-14	9
7	3.6031e-10	41	1.1698e-13	24
8	3.7478e-10	47	1.6755e-13	29
9	7.4012e-10	79	1.7597e-12	61
10	7.3727e-10	88	2.1807e-14	80

Table 6: $k = \sqrt{25.5}$

m	Error by CG	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	1.3714e-14	4	3.7447e-14	3
4	1.4021e-14	4	3.2229e-14	3
5	4.9569e-13	11	1.3548e-14	9
6	7.0131e-13	13	1.9299e-14	9
7	1.7029e-11	32	1.2026e-13	23

8	1.8469e-11	35	6.7194e-14	29
9	9.0019e-11	105	1.9202e-13	70
10	1.1680e-10	115	4.4064e-13	81

Table 7: $k = \sqrt{52}$

m	Error by CG	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	4.7981e-13	5	1.3287e-14	3
4	4.7992e-13	5	1.5014e-14	3
5	2.8706e-12	10	1.2184e-14	11
6	2.7447e-12	10	2.0203e-14	9
7	6.1810e-12	30	1.4013e-13	23
8	6.8702e-12	31	8.2519e-15	28
9	8.0517e-10	157	9.9053e-13	69
10	7.9356e-10	176	6.2127e-13	80

Table 8: $k = \sqrt{100}$

m	Error by CG	Iteration	Error by CGLS	Iteration
2	-	-	-	-
3	1.2683e-13	4	8.4558e-15	3
4	1.2539e-13	4	8.8627e-15	3
5	4.5125e-13	10	8.1817e-15	9
6	4.2888e-13	9	8.3129e-15	9
7	1.1589e-12	32	8.4087e-14	23
8	1.4781e-12	36	1.1685e-13	23
9	3.726e-11	128	2.7375e-15	66
10	4.3305e-11	159	4.0762e-13	74

5.2 non-Polynomial exact solution

In the following we study a Cauchy problem of modified Helmholtz equation with a non-polynomial exact solution.

Example 2:

We consider the Cauchy problem for a modified Helmholtz equation with exact solution $u(x,y) = exp\,(x)\,cos(y)$, defined in an annular domain with the constant radius $\rho_{e=}1$ and $\rho_i = 0.5$ and $\beta = 2$. This problem is over–specified on the outer boundary $\Gamma_1 = \{(x,y): x^2 + y^2 \ = 1\}$ for which we have the following Cauchy data $h = exp\,(x)$

$cos\ (y)\,, g = exp\ (x)\ cos\ (y)\ cos\ (\theta) - exp\ (x)\, sin(y), sin(\theta)$. We study different cases for a different physical parameter $= \sqrt{100}, \sqrt{52}, \sqrt{25.5}, \sqrt{15}$. For the numerical computations, we take $n_1 = 100\ nr = 20$ and so $n_2 = 2000$ and we take m=2, 3, …,16 We compare the results obtained by using the both algorithms CGM and CGLS with $tol = 10^{-12}$.

The obtained result is presented in tables 9,10,11,12.

Table 9: $k = \sqrt{15}$

m	Error by CG	Iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0140	4	0.0140	5
4	0.0018	7	0.0018	9
5	0.0147	13	0.0147	17
6	0.0148	18	0.0148	29
7	0.0124	29	0.0124	56
8	0.0124	46	0.0124	119
9	0.0122	77	0.0132	219
10	0.0122	149	0.0131	375
11	0.0067	325	0.0125	489
12	0.0066	600	0.0130	380
13	0.0248	1562	0.0128	502
14	0.0248	3042	0.0128	513
15	0.0644	9110	0.0130	517
16	0.064481	23930	0.01286	800

Table 10: $k = \sqrt{25.5}$

M	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0140	4	0.0140	5
4	0.0018	7	0.0018	9
5	0.0059	14	0.0059	17
6	0.0060	9	0.0060	33
7	0.0049	31	0.0049	63
8	0.0049	52	0.0049	126
9	0.0057	79	0.0058	210
10	0.0057	144	0.0058	248
11	0.0041	316	0.0059	348
12	0.0041	584	0.0061	288
13	0.0074	1483	0.0058	484
14	0.0074	3150	0.0062	448
15	0.0451	9154	0.0060	495
16	0.0451	25428	0.0059	554

Table 11: $k = \sqrt{52}$

m	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0139	4	0.0139	5
4	0.0017	7	0.0017	9
5	0.0011	14	0.0011	19
6	0.0011	21	0.0011	38
7	9.9650e-04	38	9.9650e-04	78
8	9.9121e-04	57	9.9644e-04	132
9	0.0018	93	7.9390e-04	166
10	0.0018	175	8.1003e-04	208
11	0.0012	330	0.0015	197
12	0.0012	595	0.0015	210
13	8.1622e-04	1421	0.0017	252
14	8.1145e-04	2990	0.0017	281
15	0.0138	9094	0.0018	234
16	0.01376	25655	0.00174	392

Table 12: $k = \sqrt{100}$

m	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0139	4	0.0139	5
4	0.0017	8	0.0017	9
5	2.6773e-04	13	2.6773e-04	20
6	2.0941e-04	20	2.0941e-04	39
7	0.0015	40	0.0015	91
8	0.0015	74	0.0015	167
9	0.0012	122	4.3682e-04	170
10	0.0012	222	4.4016e-04	174
11	2.0555e-04	407	3.6273e-04	181
12	1.9088e-04	782	3.0316e-04	219
13	1.2515e-04	1578	1.9804e-04	198
14	1.2472e-04	2797	1.8389e-04	211
15	0.0022	11000	3.2415e-04	228
16	0.0022	18477	3.1827e-04	222

The influence of the number n when increases 0n the accuracy is explored in table 2 where we take $n_1 = 400$, $nr = 20$, $n_2 = 8000$,and compare CG and CGLS for different n, the results are presented in tables (13,14,15,16), we observe that from m≥ 12, with the cases k =($\sqrt{15}, \sqrt{52}, \sqrt{25.5}$) and for m ≥ 10 with the case k= $\sqrt{100}$, the CGLS is very accurate and when n_1, n_2 is increases.

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Table 13: $k = \sqrt{15}$

m	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0140	4	0.0140	5
4	0.008	7	0.0018	9
5	0.0123	13	0.0123	17
6	0.0124	18	0.0124	30
7	o.o125	29	0.0125	55
8	0.0124	45	0.0124	115
9	0.0128	82	0.0132	237
10	0.0128	148	0.0132	299
11	0.0111	285	0.0132	345
12	0.0111	581	0.0130	589
13	0.0039	1446	0.0131	435
14	0.0039	3065	0.0130	512
15	0.0338	7906	0.0132	518
16	0.0338	17478	0.0132	768

Table 14: $k = \sqrt{25.5}$

m	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0139	4	0.0139	5
4	0.0018	7	0.0018	9
5	0.0034	13	0.0034	19
6	0.0035	19	0.0035	33
7	0.0048	31	0.0048	61
8	0.0049	50	0.0049	103
9	0.0059	84	0.0059	181
10	0.0059	143	0.0059	242
11	0.0054	281	0.0060	271
12	0.0054	542	0.0060	386
13	0.0032	1239	0.0059	406
14	0.0032	2728	0.0061	484
15	0.0121	8127	0.0060	429
16	0.0121	17978	0.0060	488

Table 15: $k = \sqrt{52}$

m	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0139	4	0.0139	5
4	0.0017	7	0.0017	9
5	4.5876e-04	14	4.5876e-04	19
6	4.3632e-04	20	4.3632e-04	39
7	7.1199e-04	38	7.1199e-04	82
8	7.2205e-04	59	7.2059e-04	146

9	0.0018	101	7.8251e-04	195
10	0.0018	186	7.8931e-04	206
11	0.0031	317	0.0014	191
12	0.0013	556	0.0014	178
13	0.0011	1228	0.0016	229
14	0.0011	2566	0.0016	235
15	0.0014	7574	0.0017	372
16	0.0014	16950	0.0017	361

Table 16: $k = \sqrt{100}$

m	Error by CG	iteration	Error by CGLS	iteration
2	0.0842	2	0.0842	2
3	0.0140	4	0.0140	5
4	0.0017	7	0.0017	9
5	0.0123	13	0.0123	17
6	8.0231e-05	20	8.0230e-05	39
7	0.0125	29	0.0125	55
8	7.7900e-04	73	2.0672e-04	118
9	0.0128	82	0.0132	237
10	0.0011	217	2.8844e-04	167
11	0.0111	285	0.0132	345
12	1.8508e-04	731	1.2771e-04	241
13	0.0039	1446	0.0131	435
14	2.6763e-04	3274	2.1609e-04	197
15	0.0338	7906	0.0132	518
16	1.5753e-04	14078	2.3777e-004	345

5.3 Stability and effect of a noise

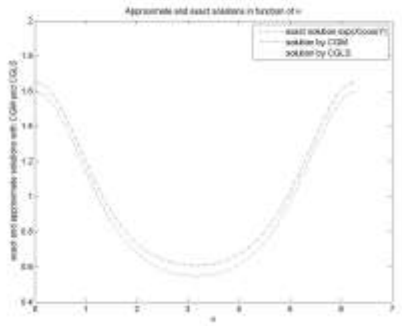
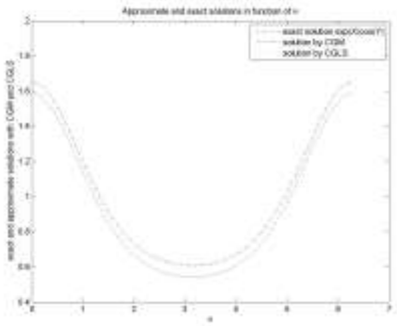
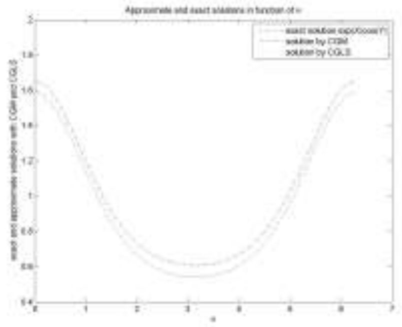
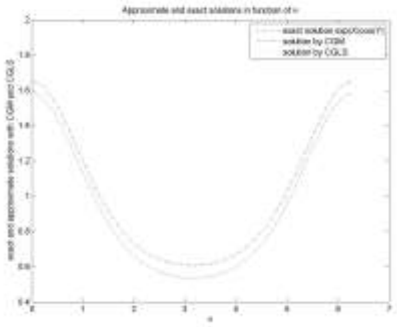
The inverse problem is a kind of problem that affected by the collected (measured) data, and since this data can have error due to measurement error. Therefore, it is important to study the effect of data noise on the approximate solution. For this, we set the noise on the Cauchy data in the form:

$$\boldsymbol{h(\theta) = u_{ex}(\rho, \theta) + \sigma * rand}$$

For some deviation of measurement errors, $\sigma = 0.1, 0.01, 0.05, 0.001$ and for a random Gaussian error $rand$. We study the perturbation of Cauchy data by a noise for example 2, for a physical parameter $\sqrt{52}$, $n1 = 200$,$n2 = 4000$ with $Tol = 10^{-12}$

σ	No. of Iteration for CGM	Error with CGM	No. of Iteration for CGLS	Error with CGLS
0.1	8913	0.051775290636955	516	0.013225283211797
0.01	9030	0.058300372536751	509	0.013099911906732
0.05	8980	0.059153986017208	511	0.013083553955267
0.001	9065	0.064337147820386	481	0.012984203670260

This table show that the number of iterations are augment a little for the CGLS and the error is augment by 0.001 for $\sigma = 0.1$ and for CGM the error is reduced by 0.01 and the number of iteration is reduced for $\sigma = 0.1$, in fact the error without noise for the CGM is equal to 0.064449922780489 with 8993 iterations and the error without noise for CGLS is equal to 0.012982105320468 with 480.

Noise parameter $\sigma = 0.1$	Noise parameter $\sigma = 0.01$
	
Noise parameter $\sigma = 0.05$	Noise parameter $\sigma = 0.001$
	

The problem in example 2 is an ill-posed problem with a big condition number about 7.020387634918484e+005. The previous figures show that the exact and the approximate solutions obtained with CGM and CGLS on the inaccessible boundary $\Gamma 2$. In fact, the approximate solution are effected slightly for all value of noise. The error augment by 0.001 for the CGLS for the noise $\sigma = 0.1$. (with about the same number of iteration for $\sigma = 0.001$ and with some more iteration for $= 0.1$) and is reduced by 0.01 for the CGM for the noise $\sigma = 0.1$ (with less number of iteration for $\sigma = 0.1$ and with more iteration for $\sigma = 0.001$), the important thing is that the solution still stable for both CGM and CGLS. The approximations keep the same accuracy, even for a high value of noise until $\sigma = 0.1$ relative random parameter, indeed this problem has a big condition number for this it is highly ill-condition.

Conclusion

We solve the inverse Cauchy problem of the modified Helmholtz equation on an annular domain for recovering unknown data on a part of the boundary from the given data on another accessible part. The inverse Cauchy problem is transformed to solve a direct problem using a polynomial expansion of the solution which implies producing a linear system and solving this system by (CGM) and (CGLS). This proposed method is verified by solving some examples and comparing the accuracy of (CGM) and (CGLS) to show that the method can overcome ill-posedness of the inverse Cauchy problem. The stability of the method is investigated by applying noise on the Cauchy data.

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