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Numerical Method for solving an inverse Cauchy problem for Helmholtz equation using CGLS, BICGSTAB and PCG

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Abstract

In this work, an inverse Cauchy problem of the Helmholtz equation for thermal conductivity at the edge of the target was considered. In this paper, the temperature on the unknown boundary (the inner boundary) is determined by taking advantage of the Cauchy data that can be obtained from the known part, the part that can be reached (the outer boundary), This problem is solved numerically using the proposed method, and the stability of the approximate solution has been confirmed by adding noise to the Cauchy data.

Key words: inverse Cauchy problem, modified Helmholtz equation, polynomial expansion, Conjugate Gradient Least Square Method (CGLS), Bi-Conjugate Gradients Stabilized Method (BICGSTAB), Preconditioned Conjugate Gradients Method (PCG).

الطرق العددية لحل مسألة كوشي العكسية لمعادلة هيلمهولتز باستخدام *CGLS* , BICGSTAB و PCG

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الخلاصة

في هذا العمل ، تم النظر في مشكلة كوشي العكسية لمعادلة هيلمهولتز للتوصيل الحراري عند حدود الهدف. في هذا البحث ، سنحدد درجة الحرارة على الحدود غير المعروفة (الحد الداخلي) ، من خلال الاستفادة من بيانات كوشي التي يمكن

الحصول عليها من الجزء المعروف، الجزء الذي يمكن الوصول إليه (الحد الخارجي). تم حل هذه المشكلة عدديًا باستخدام الطريقة المقترحة ، وتم التأكد من ثبات الحل التقريبي بإضافة ضوضاء لبيانات كوشي.

الكلمات المفتاحية: مشكلة كوشي العكسية ، معادلة هيلمهولتز المعدلة ، توسع متعدد الحدود ، طريقة التربيع الصغرى المترافقة المتدرجة (CGLS) ، طريقة التدرجات المتقاربة المتقاربة (BICGSTAB) ، طريقة التدرج المتقارن المشروطة مسبقًا (PCG)

Introduction

Finding an unknown boundary and its resistive properties is one of the crucial applications in inverse problem design and optimization. This encourages us to solve an inverse Cauchy problem governed by a Helmholtz equation to determine the temperature on appear-shaped inner border inside an ellipse form domain.

In this article, we take a look at the inverse problem, which entails estimating the temperature t on an inner border profiting from the given Cauchy data on the outer boundary (boundary temperature and heat flux). Assuming that the steady-state temperature t satisfies the Helmholtz equation governing thermal conductivity at the target edge:

$$\nabla^2 t + k^2 t = 0 \qquad \Omega\backslash D$$

Based on the knowledge of the Dirichlet temperature data t and Neumann heat flux data $\frac{\partial t}{\partial n}$ on the outer part of the boundary $\partial\Omega$ where n is the outward unit normal at $\partial\Omega$, and a boundary condition (Dirichlet, Neumann or Robin) on the boundary ∂D of D [18]. These kinds of problems are ill-posed. In reality, in Hadamard's view, a problem is well-posed if the existence, uniqueness, and stability of the solution are guaranteed [7]. Otherwise, the problem is ill-posed if the solution does not meet one of these requirements, in which case an inverse problem must be developed to address it. The inverse problem is typically thought to be more challenging to solve than the direct ones.

In addition, the inverse problems are unstable [7], meaning that even a minor measurement mistake in the input data might result in a significant inaccuracy in the solution. Inverse Problems have recently been discussed in a number of scientific fields see Kubo, 1988 [16]. One of the inverse problem examples is the Cauchy problem. Chakib et al., 2018 [6], Hernandez-Montero et al., 2019 [8], Isakov, 2017[12], Kabanikhin, 2012 [15], Lavrent'ev, 1986 [17], Liu and Wang, 2018 [19], and Nachaoui et al., 2021 [21] are a few references to this. When it comes to these kinds of problems, the boundary conditions (Dirichlet, Neumann) are only known for a portion of the boundary (the accessible portion), while the other portion of the boundary lacks any available information, making it under-specified or inaccessible.

To address this kind of problems, an appropriate method must be selected in order to lessen the ill-posedness of the studied problem. Numerous techniques have been developed during the past 20 years to solve the Cauchy problem for the Helmholtz equation. The truncation method Yang, 2019 [26], the conjugate gradient method Marin et al., 2003 [20], the meshless generalized finite difference method Hua et al., 2017 [9], and the fractional Tikhonov regularization method Qia and Feng, 2017[23] are a few of these methods that are used for this type of problems.

In reality, the quality of approximation is significantly impacted by the reliance of the numerical solutions to the direct Helmholtz equation on the physical parameter; for additional information, see Ihlenburg and Babuka, 1995 and 1997 [10][12]. See [3], [4], [5], and [23] for several approaches that have been developed to solve the Cauchy Helmholtz equation for some large parameter. Jourhmane and Nachaoui, 1996 [13] suggested an alternating algorithm based on relaxation of alternating algorithms.

According to Berdawood et al., 2021[2], an efficient relaxed alternating procedure proved the convergence for all values of wave numbers k in the case of the Helmholtz equation and accelerated the convergence in the case of the modified Helmholtz equation. They also demonstrated that we can find an interval of relaxation parameter in which the convergence is guaranteed for any value wave number k.

In order to approximate the solution of a Cauchy Problem for Helmholtz-type equation in a confined domain surrounded by a smooth boundary, the goal of this work is to investigate an approach based on polynomial expansion. In this study, a meshless method to approximate the temperature on the inaccessible inner boundary, following the method proposed by Rasheed et al. [24] in which they use it to solve an inverse Cauchy problem for Laplacian equation. This approach was also used by Jameel et al. in 2022 [14] and by Jameel [14]. The rest of this paper is structured as follows. Section 2 reminds us of the inverse Cauchy problem for the Helmholtz equation. Section 3 of our paper provides our proposed approximation approach. In section 4, a few numerical techniques are examined and shown by using them for chosen cases.

Inverse Cauchy problems for the Helmholtz equation

The domain $\Omega \subset \mathbb{R}^2$ with the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$

$$\Gamma_1 = \{(\mathrm{r},\theta) : r = \rho_e(\theta) \quad 0 \leq \theta \leq 2\pi\} \text{ outer boundary}$$

and

$$\Gamma_2 = \{(\mathrm{r},\theta) : r = \rho_i(\theta) \quad 0 \leq \theta \leq 2\pi\} \text{ inner boundary}$$

where $0 < \rho_e(\theta) \leq 1,\ 0 < \rho_i(\theta) < 1.$

We take into account the inverse Cauchy problem for the Helmholtz equation, which is as follows

$$\Delta t(x,y) + k^2 t(x,y) = F \qquad \Omega\backslash D \tag{1}$$

$$t(\rho,\theta) = h(\theta) \quad on\ \partial\Omega = \Gamma_1 \tag{2}$$

$$\frac{\partial t}{\partial n}(\rho,\theta) = g(\theta) \quad on\ \partial\Omega = \Gamma_1 \tag{3}$$

Take note that F is given on $\Omega\backslash D$ and the accessible portion of the domain border contains the Cauchy data $t(x,y)$ and $\partial_n t(x,y)$, Say $h(\theta), g(\theta)$ respectively. Two boundary criteria are stated for the part Γ_1. While no boundary condition is stated on Γ_2. To ascertain the temperature t on the interior under-determined boundary, the inverse problem for the modified Helmholtz equation is formulated. Remembering that the following expression may be used to define the normal derivative of , denoted by $\partial_n t$ from [20] :

$$\partial_n t(\rho,\theta) = \eta(\theta)\left[\frac{\partial t(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial t(\rho,\theta)}{\partial \theta}\right] \tag{4}$$

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta)+[\rho'(\theta)]^2}} \tag{5}$$

The normal derivative $\partial_n t(x, y)$ can be expressed in terms of $\partial_x t$ and $\partial_y t$ by

$$\partial_n t = \eta(\theta)\left[\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right]\partial_x t + \eta(\theta)\left[\sin(\theta) - \frac{\rho'}{\rho^2}\cos(\theta)\right]\partial_y t \tag{6}$$

Approximation of Solution by a Polynomial Expansion

The solution $t(x, y)$ is expressed as a polynomial expansion as follows:

$$t(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij}\, x^{i-j} y^{j-1} \tag{7}$$

To find $t(x, y)$, the coefficients c_{ij} must be determined the number of these coefficient is $n = \frac{m(m+1)}{2}$, and the maximal order of above polynomial is $m - 1$. Using equation (7) we find $\partial_x\, t(x, y)$, $\partial_y\, t(x, y)$, and Δt :

$$\partial_x\, t(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij}\,(i-j)\, x^{i-j-1} y^{j-1} \tag{8}$$

$$\partial_y\, t(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij}\,(j-1)\, x^{i-j} y^{j-2} \tag{9}$$

$$\partial_{xx}\, t = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij}\,(i-j)(i-j-1)\, x^{i-j-2} y^{j-1} \tag{10}$$

$$\partial_{yy}\, t(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij}\,(j-1)(j-2) x^{i-j} y^{j-3} \tag{11}$$

$$\Delta t(x, y) + k^2 u(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{i} [c_{ij}(i-j)(i-j-1)\, x^{i-j-2} y^{j-1} +$$

$$(j-1)\,(j-2) x^{i-j} y^{j-3}] + k^2 x^{i-j} y^{j-1} \tag{12}$$

The coefficient c_{ij} in equation (7) is shown as an n-dimensional vector c with component c_k where k=1,...,n. In reality, the coefficients c_{ij} are reordered taking into account that $i = 1, \dots, m$, $j = 1, \dots, i$ for each index i, j. Assuming the formula $k = \frac{i(i-1)}{2} + j$ to correspond to one index k. The vector t is given by the inner product of a^T with c.

$$t(x,y) = [1\ x\ y\ x^2\ xy\ y^2\ \ x^3 x^2 y\ \ xy^2\ \ y^3\ \ \ldots]\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = a^T \tag{13}$$

replace(8) and (9) in to (6) gives us an expression of $\partial_n t$. Similarly, for each point on the accessible part of the boundary Γ_1 the normal derivative $\partial_n t$ can be expressed as an inner product of a vector e with c, such that the component $\imath - th$ of e is given by:

$$e_\imath = \eta(\theta)\left[(i-j)x^{i-j-1}y^{j-1}\left(\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right) + (j-1)x^{i-j}y^{j-2}\left(\sin(\theta) - \frac{\rho'}{\rho^2}\cos(\theta)\right)\right] \tag{14}$$

For $\imath$ =1, …, n and by keeping the same indices i, j of the coefficients for those used to calculate $e_\imath$from $c_{ij.}$ Now for each point in the domain the term $\Delta u(x,y) - k^2 u(x,y)$ can be expressed from (12) as an inner product of a vector d with c, where the $\imath - th$ component $d_\imath$, $\imath$=1, …,n and is given by:

$$d_\imath = (i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} - k^2 x^{i-j}y^{j-1} \tag{15}$$

Choosing n_1 points on boundary Γ_1 , say $(x_{i,}y_i) = (\cos(\theta_i), \sin(\theta_i))$, $i = 1,2, \ldots n_1$ to verify the boundary condition (2)-(3) and also we take n_2 points in the domain $\Omega \backslash D$, say $(x_{j,}y_j)$ $j = 1,2, \ldots n_2$, to satisfy the equation (1), So we obtain the a linear system

$$Ac = b \tag{16}$$

The vector b is of longer $(2n_1 + n_2)$ and A is $((2n_1 + n_2) \times \frac{m(m+1)}{2})$ matrix given respectively by:

$$\mathrm{A} = [a_1^T \cdots a_{n1}^T\, e_1^T \cdots e_{n1}^T\, d_1^T .. d_{n2}^T], b = [h(\theta_1)\ \cdots h(\theta_{n1}) g(\theta_1) \cdots g(\theta_{n1})\ F_1 \cdots F_{n_2}\ ] \tag{17}$$

Solving the linear System

To solve the linear system $Ac = b$, we use the well-known Conjugate Gradient least square method (CGLS) [1], Bi-conjugate Gradients Stabilized method (BICGSTAB) [25] and Preconditioned Conjugate Gradients Method (PCG)[1].

Stopping criterion

The condition under which the algorithm can stop is crucial for any numerical approach, thus we selected the following stopping criteria

$$Absolute\ error\ < Tol \tag{18}$$

$$Relative\ error < Tol \tag{19}$$

Additionally, we must specify the beginning data for the algorithms CGLS, BICGSTAB and PCG. To do this, we provide an educated approximation for t on the under –specified boundary Γ_2 and in the initial iteration, the zero vector was used as an initial guess.

Numerical Results and Discussion

In this part, we examine a few cases and numerically resolve them to demonstrate the effectiveness of our suggested approach. Considering two cases of exact solution (polynomial and non-polynomial), this exact solution is used to calculate the function F, its trace h and its normal derivative g on Γ_1. Utilizing the precise supplied data on the accessible portion of the boundary Γ_1, zero beginning data for solving the linear system using CGLS, BICGSTAB and PCG like-methods mentioned in sections 6.1 and 6.2, with appropriate tolerance and stopping criteria.

1. **Polynomial case**

In the following we study a Cauchy problem of Helmholtz equation with some polynomial exact solution.

Example 1: Consider the Cauchy problem for a Helmholtz equation with exact solution$t(x,y) = 6x^2y^2 - x^4 - y^4$, defined by an ellipse form domain with a pear-shaped inner boundary $\rho_{e=}(0.5 * 0.4)/sqrt(0.25 * (\cos(\theta))^2 + 0.16 * (\sin(\theta))^2)$ and $\rho_i = 0.6 + 0.125 * \cos(3 * \theta)$ i.e $\Gamma_1 = \{(x,y) := 0.6 + 0.125 * \cos(3 * \theta)\}$. With the following Cauchy data: $h = 6x^2y^2 - x^4 - y^4$,

$$g = \eta(\theta)\left[cos\ (\theta)\ - \frac{\rho'}{\rho^2} sin\ (\theta)\right]\partial_x t + \eta(\theta)\left[sin\ (\theta)\ - \frac{\rho'}{\rho^2} cos\ (\theta)\right]\partial_y t$$

where ρ changes at every θ. We investigate many values for a different physical parameter. $k = \sqrt{15}, \sqrt{25.5}, \sqrt{52}$. For the numerical calculations, we take:

$$n_1 = 63, nr = 5, \text{ so } n_2 = 315, \text{ and } m = 2, \ldots, 10$$

contrasting the approximate solution produced by applying the CGLS, BICGSTAB and PCG algorithms with $tol = 10^{-12}$. The obtained results are presented in tables 1,2,3

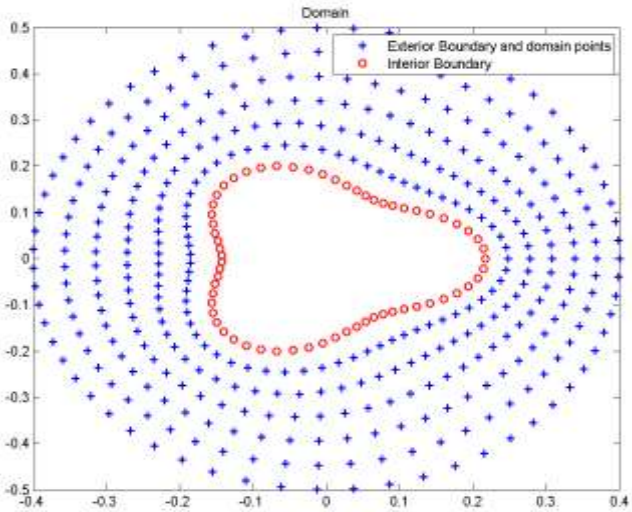

Figure 1: The Domain for Example 1

Table 1: Results by CGLS, BICGSTAB and PCG for $k = \sqrt{15}$ with $\beta = 2$, different value of m and $Tol = 10^{-12}$

m	Iteration cgls	Relative error cgls	Iteration bicgstab	Relative error bicgstab	Iteration pcg	Relative error pcg
2	2	9.69532919E-01	1.5	9.69532919E-01	2	9.69532919E-01
3	5	3.39222809E+00	4.5	3.39222809E+00	5	3.39222809E+00
4	9	9.10773081E+00	10.5	9.10773081E+00	9	9.10773081E+00
5	**18**	**9.85279080E-13**	**25.5**	**3.35070376E-11**	**18**	**1.95296785E-11**
6	30	2.81320132E-11	90.5	1.53611473E-10	30	2.29419799E-10
7	60	3.67254071E-12	406.5	6.14751311E-10	63	1.92803179E-09
8	108	5.46202437E-10	2527.5	6.21897883E-10	121	4.99766518E-10
9	273	1.18365009E-12	24335	1.92979054E-02	284	3.56822472E-10
10	592	3.86602847E-10	8874	4.37307360E-04	651	7.77403825E-10

For m=5 the relative error is equal **9.85279080E-13**for CGLS, **3.35070376E-11** for BICGSTAB and **1.95296785E-11**for PCG that is a good approximation .The mistakes and comparison between the precise solution and the approximation obtained by CGLS, BICGSTAB, and PCG are shown in the figures below.

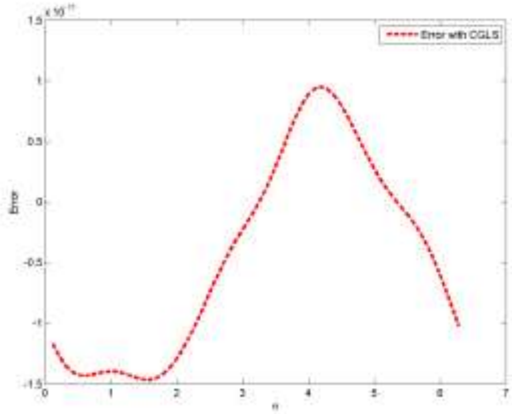

Figure 2.a: Error by CGLS

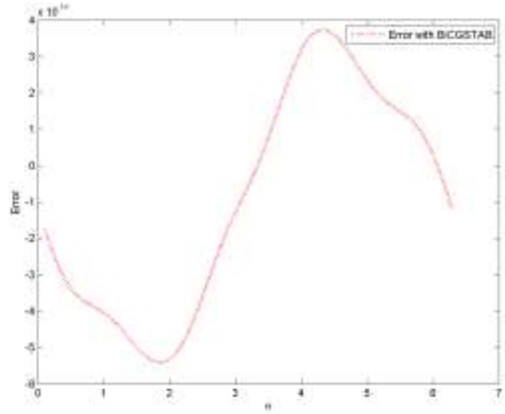

Figure 2.b: Error by BICGSTAB

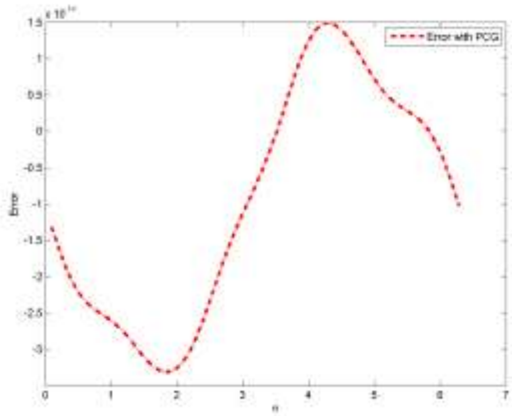

Figure 2.c: Error by PCG

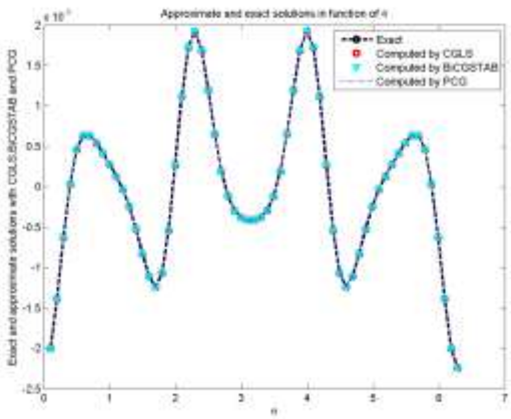

Figure 2.d: Comparison of exact and approximate solutions

Now, we give the results for $k = \sqrt{25.5}$:

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Table 2: Results by CGLS , BICGSTAB and PCG for $k = \sqrt{25.5}$ with $\beta = 2$, different value of m and $Tol = 10^{-12}$

m	iteration CGLS	Relative Error CGLS	iteration BICGSTAB	Relative Error BICGSTAB	iteration PCG	Relative Error PCG
2	2	9.69420047E-01	1.5	9.69420047E-01	2	9.69420047E-01
3	5	1.33340696E+00	4.5	1.33340696E+00	5	1.33340696E+00
4	9	6.90992732E+00	9	6.90992732E+00	9	6.90992732E+00
5	**20**	**1.49603744E-13**	**31**	**1.96529893E-10**	**20**	**4.00257780E-12**
6	39	1.30796757E-12	173	1.05715939E-09	38	7.94199652E-10
7	65	2.70963175E-10	731.5	1.51869797E-10	77	1.75857159E-11
8	125	1.10574178E-09	7383	2.68053074E-08	145	3.32221169E-09
9	274	2.91417943E-09	11057	6.52970915E-04	306	3.35635603E-09
10	683	1.00323558E-09	3018	6.28593345E-03	744	8.68182686E-10

For m=5 the relative error is equal **1.49603744E-13** for CGLS, **1.96529893E-10** for BICGSTAB and **4.00257780E-12** for PCG that is a good approximation .The errors and comparison between the precise solution and the approximation obtained by CGLS, BICGSTAB, and PCG are shown in the figures below.

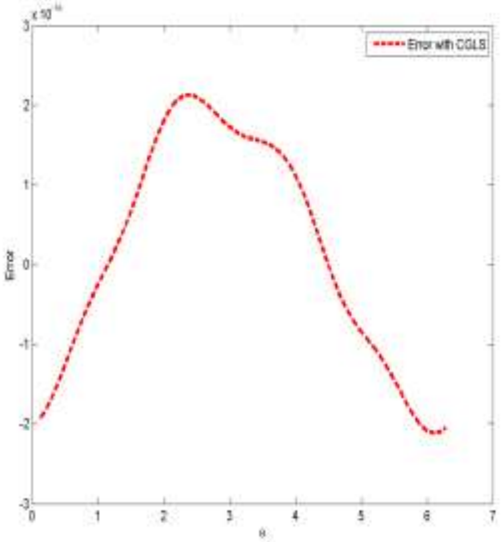

Figure 3.a: Error by CGLS

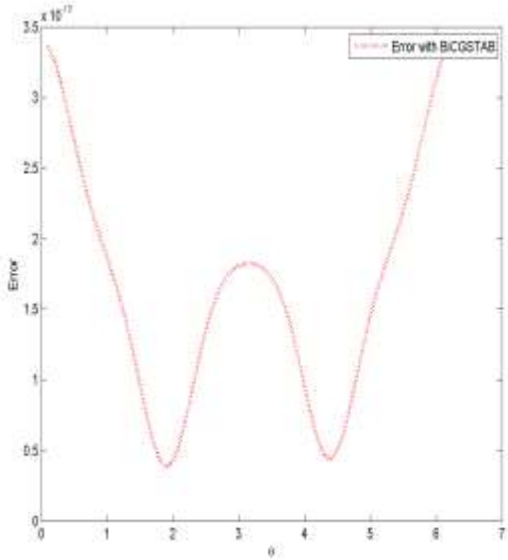

Figure 3.b: Error by BICGSTAB

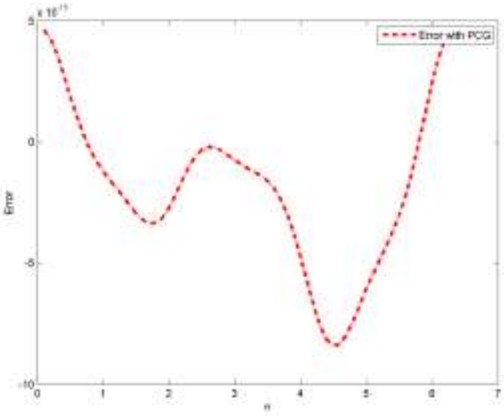

Figure 3.c: Error by PCG

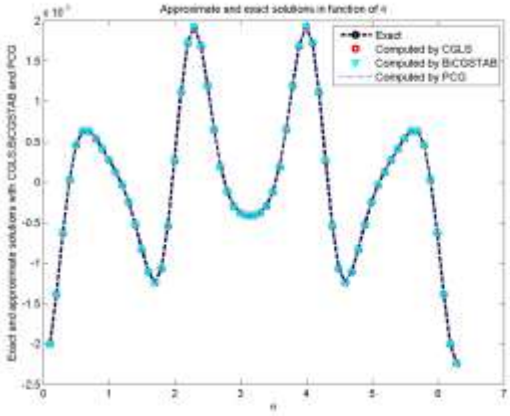

Figure 3.d: Comparison of exact and approximate solutions

Now, we give the results for $k = \sqrt{52}$:

Table 3: Results by CGLS, BICGSTAB and PCG for $k = \sqrt{52}$ with $\beta = 2$, different value of m and $Tol = 10^{-12}$

m	iteration CGLS	Relative Error CGLS	iteration BICGSTAB	Relative Error BICGSTAB	iteration PCG	Relative Error PCG
2	2	9.69379643E-01	1.5	9.69379643E-01	2	9.69379643E-01
3	5	1.38826578E+00	5.5	1.38826578E+00	5	1.38826578E+00
4	11	4.23033843E+00	9	4.23033843E+00	9	4.23033843E+00
5	**21**	**4.35753252E-12**	**31**	**5.12507238E-11**	**20**	**4.11563815E-11**
6	44	3.26990410E-11	333	5.95615615E-09	48	1.03247614E-10
7	84	7.85962000E-12	4863	5.59334862E-02	92	3.07710603E-09
8	149	3.03336161E-09	22230	2.74194244E-04	167	1.30555108E-08
9	303	4.30665449E-08	4770	8.23429153E-02	367	1.25260905E-08
10	597	1.27516694E-07	18948	2.71179222E-02	776	1.24895335E-07

For m =6 the relative error is 1.49539075E-13 for CGLS and for m =5 the relative error is 2.26265781E-11 for BICGSTAB and for m = 5 the relative error is 6.47774081E-13 for PCG which is an estimate Good approximation,

Tables 1,2,3 show that the best accuracy for the approximate solutions are obtained for m $=$ 5 and this comes from the fact that we approximate a polynomial of degree 4 with a polynomial of degree 4, so it is normal to obtain the best accuracy for m = 5 and that the number of iterations and the accuracy for CGLS and PCG are importantly less than BiCGSTAB.

Errors and comparison between the exact solution and the approximation obtained by CGLS, BICGSTAB and PCG are shown in the figures below:

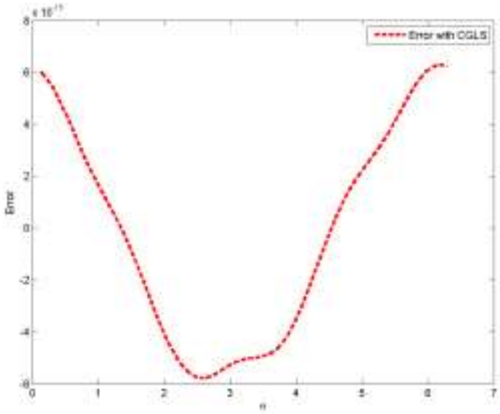

Figure 4.c: Error by CGLS

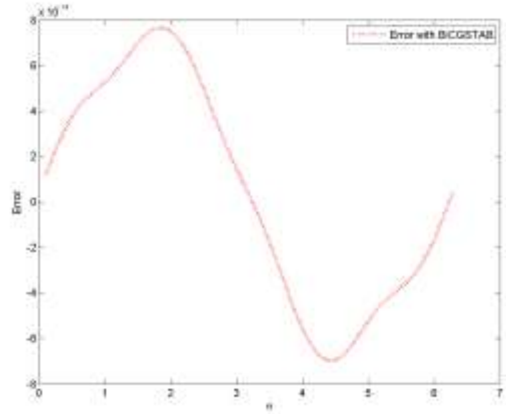

Figure 4.c: Error by BICGSTAB

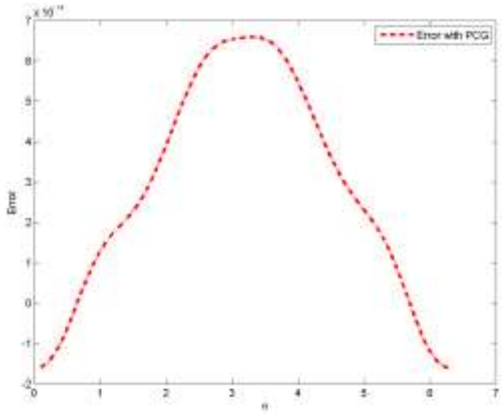

Figure 4.c: Error by PCG

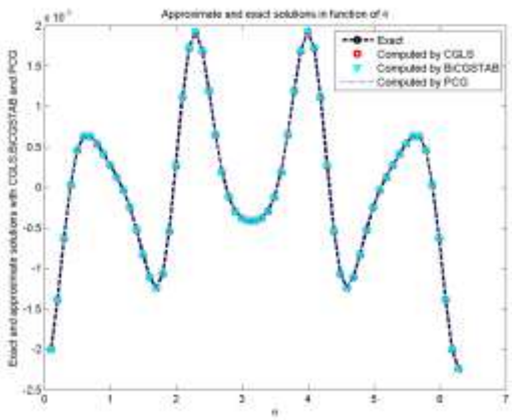

Figure 4.d: Comparison of exact and approximate solutions

2. **Non-Polynomial case**

We examine a Cauchy problem of the Helmholtz equation with a non-polynomial exact solution.

Example 2: Consider the Cauchy problem for a Helmholtz equation with exact solution $t(x,y) = \mathrm{e}^{-x^2}$, defined by an ellipse form domain with a pear-shaped inner boundary, i.e. $\rho_{e=}(0.5 * 0.4)/\sqrt{(0.25(cos(\theta)\,)^2 + 0.16(sin(\theta))^2)}$and $\rho_i = 0.6 + 0.125 * \cos(3 * \theta)$ i.e $\Gamma_1 = \{(x,y): = 0.6 + 0.125 * \cos(3 * \theta)\}$.

It has the following Cauchy information. $h = \mathrm{e}^{-x^2}$,

$$g = \eta(\theta)\left[cos\,(\theta)\; - \frac{\rho'}{\rho^2} sin\,(\theta)\right]\partial_x t + \eta(\theta)\left[sin\,(\theta)\; - \frac{\rho'}{\rho^2} cos\,(\theta)\right]\partial_y t$$

where ρ changes at every θ. we investigate many values for a different physical parameter $k = \sqrt{15}, \sqrt{25.5}, \sqrt{52}$.

For the numerical calculations, we take $n_1 = 63, nr = 5$, so $n_2 = 315$, and $m = 2, \dots, 13$ contrasting the approximate solutions produced by applying the CGLS, BICGSTAB and PCG algorithms with $tol = 10^{-12}$. The obtained results are presented in tables 4,5,6.

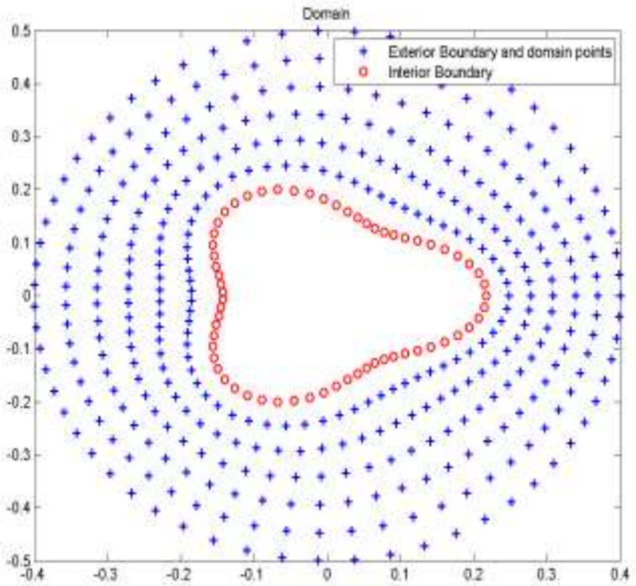

Figure 5: The Domain for Example 2

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Table 4: Results by CGLS, BICGSTAB and PCG for $k = \sqrt{15}$ with $\beta = 2$, different value of m and $Tol = 10^{-12}$

m	Iteration cgls	Relative error cgls	Iteration bicgstab	Relative error bicgstab	Iteration pcg	Relative error pcg
2	2	1.47463068E-01	1.5	1.47463068E-01	2	1.47463068E-01
3	5	3.31231238E-02	4.5	3.31231238E-02	5	3.31231238E-02
4	9	3.29642026E-02	9.5	3.29642026E-02	9	3.29642026E-02
5	18	3.22516777E-04	19	3.22516766E-04	18	3.22516777E-04
6	29	3.14273426E-04	77.5	3.14273141E-04	29	3.14273434E-04
7	49	1.22428076E-06	595.5	1.22357213E-06	55	1.22428307E-06
8	101	1.31388633E-06	2974.5	1.31395364E-06	104	1.31353407E-06
9	162	**3.18028524E-08**	7285	3.06213643E-07	183	**3.25695581E-08**
10	272	3.62244007E-08	4425	3.43652080E-07	288	3.67475545E-08
11	336	4.24446052E-08	6847	3.04400651E-07	387	4.21250819E-08
12	345	4.79378442E-08	12924	**1.16325348E-07**	400	4.75800336E-08
13	327	4.53077398E-08	5382	1.63413495E-07	382	4.47996853E-08

For m =9 the relative error is **3.18028524E-08** for CGLS and for m =12 the relative error is **1.16325348E-07** for BICGSTAB and for m = 9 the relative error is **3.25695581E-08** for PCG which is a good approximation.

Errors and comparison between the exact solution and the approximation obtained by CGLS, BICGSTAB and PCG are shown in the figures below:

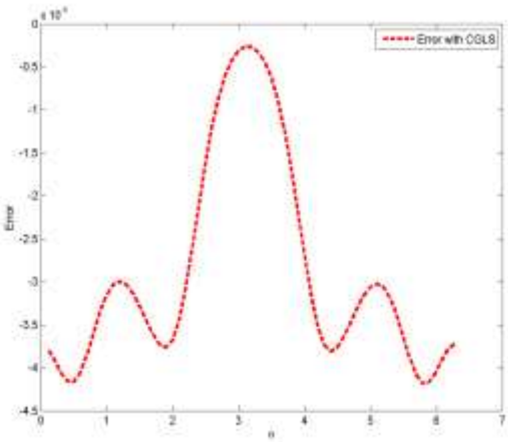

Figure 6.a: Error by CGLS

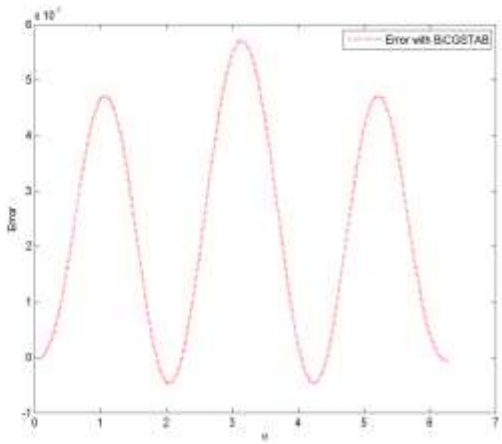

Figure 6.a: Error by BICGSTAB

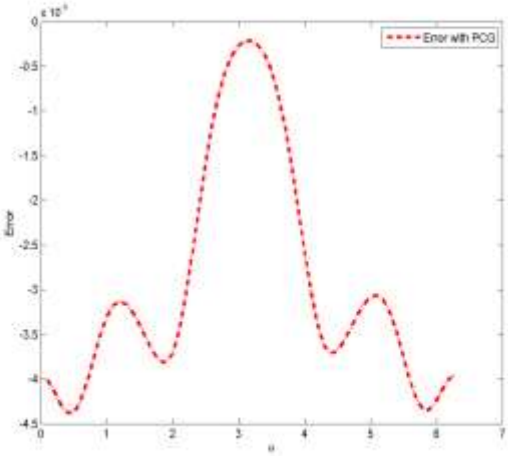

Figure 6.a: Error by PCG

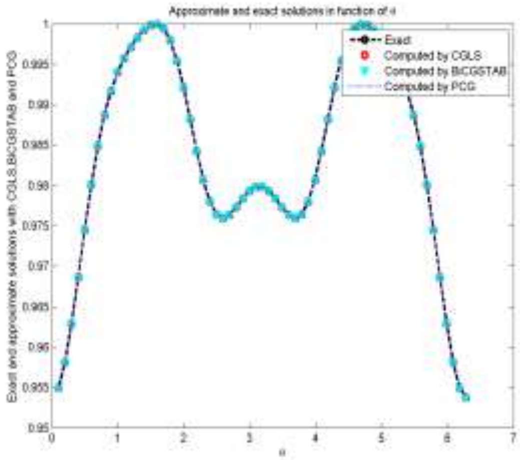

Figure 6.d: Comparison of exact and approximate solutions

Table 5: Results by CGLS, BICGSTAB and PCG for $k = \sqrt{25.5}$ with $\beta = 2$, different value of m and $Tol = 10^{-12}$

m	iteration CGLS	Relative Error CGLS	iteration BICGSTAB	Relative Error BICGSTAB	iteration PCG	Relative Error PCG
2	2	9.93673625E-02	1.5	9.93673625E-02	2	9.93673625E-02
3	5	1.58060664E-02	4.5	1.58060664E-02	5	1.58060664E-02
4	9	1.57611485E-02	8.5	1.57611485E-02	9	1.57611485E-02
5	18	5.41541405E-04	35	5.41541394E-04	20	5.41541405E-04
6	29	5.31214676E-04	141.5	5.31213317E-04	34	5.31214735E-04
7	58	2.31101958E-06	1613.5	2.31864467E-06	63	2.31072472E-06
8	96	1.75164694E-06	13378.5	1.74873071E-06	118	1.74910014E-06
9	186	**2.30112876E-08**	10879	5.76782156E-07	198	**2.44621869E-08**
10	206	3.81608613E-08	4445	5.70812873E-07	236	3.88050780E-08
11	327	6.44041262E-08	4812	**6.18768096E-07**	316	1.05660154E-07
12	322	7.17347975E-08	3409	6.22302027E-07	383	6.34466367E-08

For m =9 the relative error is **2.30112876E-08** for CGLS and for m =11 the relative error is **6.18768096E-07**for BICGSTAB and for m = 9 the relative error is **2.44621869E-08**

For PCG which is an estimate Good approximation, Errors and comparison between the exact solution and the approximation obtained by CGLS, BICGSTAB and PCG are shown in the figures below:

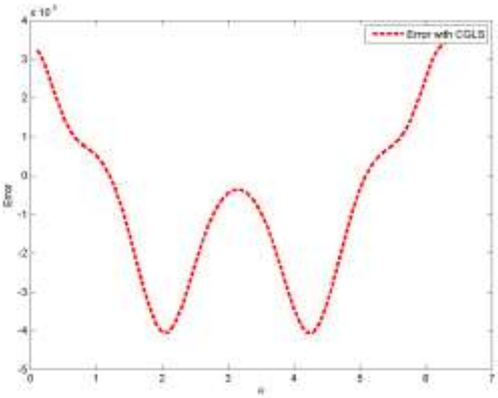

Figure 7.a: Error by CGLS

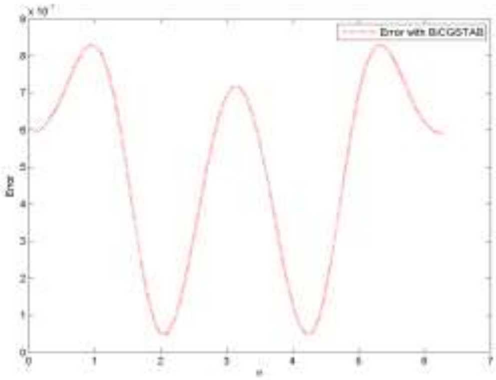

Figure 7.a: Error by BICGSTAB

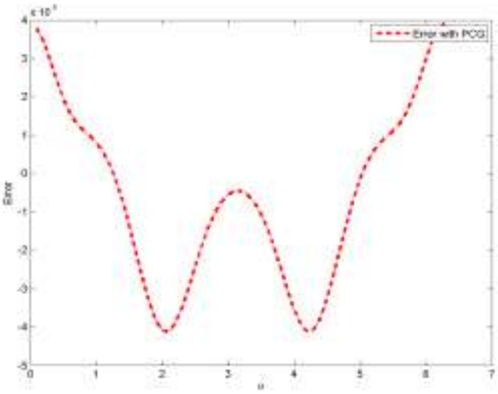

Figure 7.c: Error by PCG

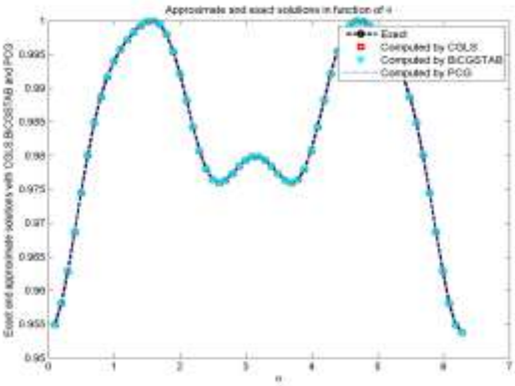

Figure 7.d: Comparison of exact and approximate solutions

Table 6: Results by CGLS, BICGSTAB and PCG for $k = \sqrt{52}$ with $\beta = 2$, different value of m and $Tol = 10^{-12}$

m	iteration CGLS	Relative Error CGLS	iteration BICGSTAB	Relative Error BICGSTAB	iteration PCG	Relative Error PCG
2	2	6.45718145E-02	1.5	6.45718145E-02	2	6.45718145E-02
3	5	5.12308178E-03	4.5	5.12308178E-03	5	5.12308178E-03
4	9	5.10957471E-03	8.5	5.10957471E-03	9	5.10957471E-03
5	19	4.59842835E-04	32.5	4.59846311E-04	20	4.59842835E-04
6	34	4.46434451E-04	379	4.46433914E-04	38	4.46434393E-04
7	71	9.56118239E-06	908	1.45930965E-04	81	9.56127017E-06
8	116	9.41077486E-06	4284	**2.43564660E-05**	132	9.41066435E-06
9	222	**5.01094248E-08**	3781	5.26420672E-05	266	**4.36082067E-08**
10	171	1.50441175E-06	4790	2.74399460E-05	273	2.03952743E-07
11	227	1.89687572E-07	2871	3.61826295E-05	268	1.84779849E-07
12	227	1.88584499E-07	2984	6.67681208E-05	271	1.76013413E-07

For m =9 the relative error is **5.01094248E-08** for CGLS and for m =8 the relative error is **2.43564660E-05** for BICGSTAB and for m = 9 the relative error is **4.36082067E-08** for PCG which is an estimate Good approximation.

Errors and comparison between the exact solution and the approximation obtained by CGLS, BICGSTAB and PCG are shown in the figures below

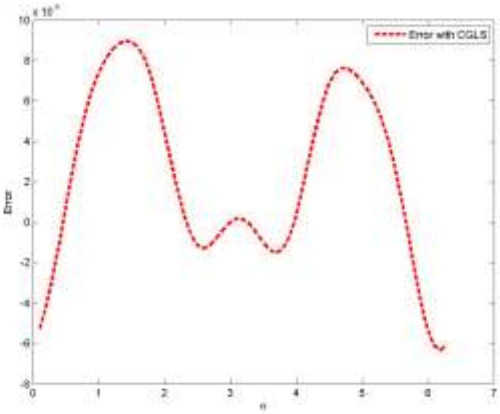

Figure 8.a: Error by CGLS

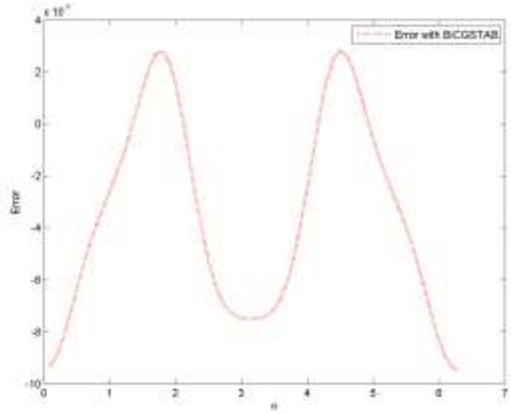

Figure 8.a: Error by BICGSTAB

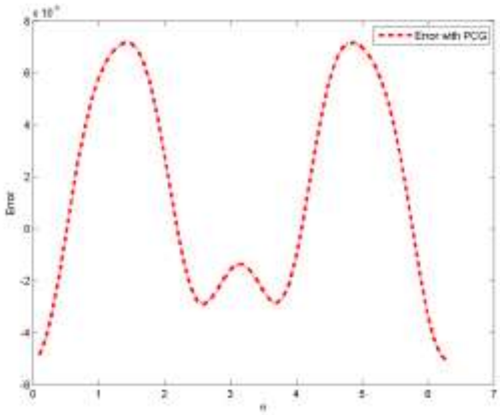

Figure 8.c: Error by PCG

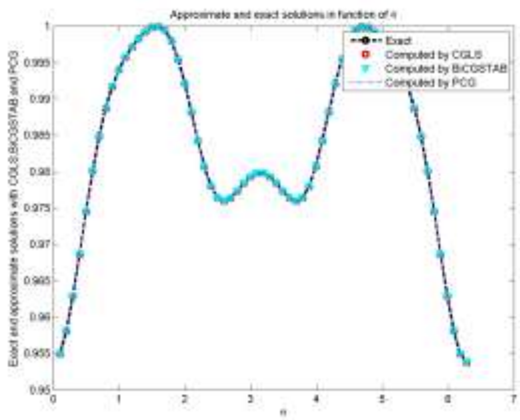

Figure 8.d: Comparison of exact and approximate solutions

Tables 4,5,6 show that the best accuracy for the approximate solutions are obtained for m $=$ 9 for CGLS and PCG while for BiCGSTAB the best accuracy obtained for m=12,11, 8, for k = $\sqrt{15}$, $\sqrt{25.5}$, $\sqrt{52}$ respectively, and that the number of iterations and the accuracy for CGLS and PCG are importantly less than BiCGSTAB.

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Stability and effect of a noise

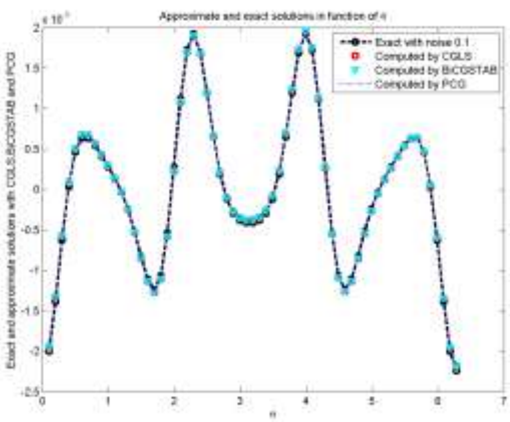
The inverse problem is a type of problem caused by the gathered (measured) data, and as these data are subject to measurement errors, they may contain errors. Therefore, it is crucial to research how data noise affects the approximation of the solution. To do this, we apply noise to the Cauchy data using the following:

$$h(\theta) = u_{ex}(\rho, \theta) + \sigma * ra$$

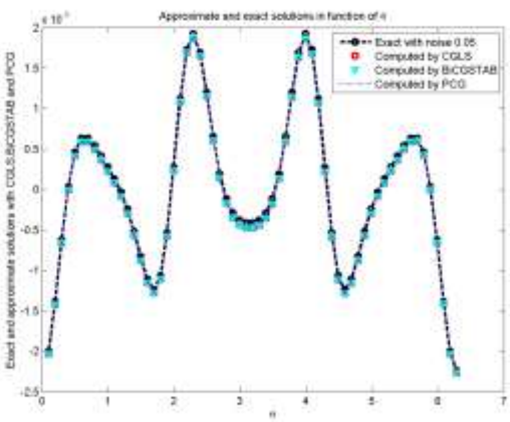
For some measurement error deviation $\sigma = 0.1, 0.01, 0.05, 0.001$ and $rand$ for a Gaussian error that is random

Table 7: effects of noise on Cauchy data, using example 1 with physical parameter $\mathrm{k} = \sqrt{52}$, $n_1 = 63, n_2 = 315$,, and $\mathrm{Tol} = 10^{-12}$.

	iteration CGLS	Relative Error CGLS	iteration BICGSTAB	Relative Error BICGSTAB	iteration PCG	Relative Error PCG
without noise	**21**	**4.35753252E-12**	**31**	**5.12507238E-11**	**20**	**4.11563815E-11**
σ=0.1	33	3.15452838E-02	61.5	3.15452846E-02	32	3.15452838E-02
σ=0.05	33	4.91792781E-02	61.5	4.91792782E-02	33	4.91792782E-02
σ=0.01	31	1.41685575E-03	57.5	1.41685573E-03	33	1.41685574E-03
σ=0.001	33	6.12181025E-04	58	6.12181299E-04	33	6.12180972E-04



Noise parameter = 0.1



Noise parameter = 0.05

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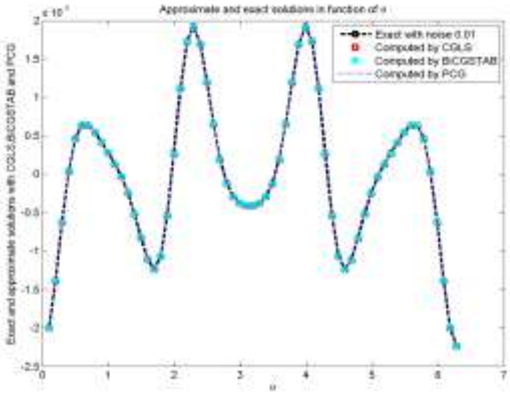
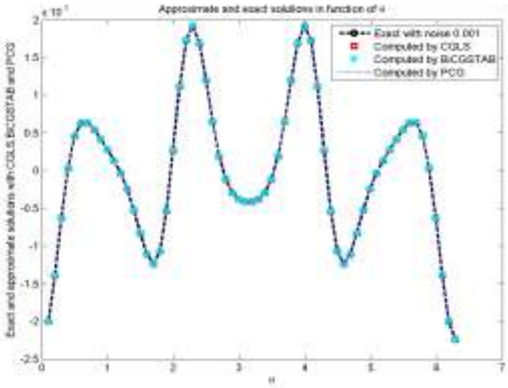


Noise parameter = 0.01



Noise parameter = 0.001

We observe that even with a high level of noise 0.1 the approximate solution still have good accuracy and have the same geometry of the exact solution, which confirm the stability of the proposed method.

Conclusion

We resolve the Helmholtz inverse Cauchy problem on an annular domain to retrieve the unknown data on the border from the supplied data on the other accessible part. The polynomial expansion of the solution, which implies to construct a linear system and solving this system by (CGLS),(BICGSTAB) and (PCG) is used to adapt the inverse Cauchy problem to solve a direct problem. In order to demonstrate that the suggested technique may circumvent the inverse Cauchy problem's ill-posedness, various cases are solved and the accuracy of (CGLS),(BICGSTAB) and (PCG) are compared. By adding noise to the Cauchy data, the stability of the approach is confirmed.

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