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Solving an Inverse Cauchy Problem for Modified Helmholtz Equations Depending on a Polynomial Expansion Approximation

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Abstract

In this study, heat conduction in fin-related inverse problem of the modified Helmholtz equation was taken into consideration. The purpose of this study is to estimate the temperature on an under-specified boundary (a part of the outer border of a given domain) by using the Cauchy data, on a portion of the boundary that is accessible (boundary temperature and heat flux). The suggested meshless approach is used to numerically solve this problem. By injecting a noise to Cauchy data, the stability is verified.

Key words: Inverse Cauchy Problem (ICP), Modified Helmholtz Equation (MHE), Polynomial Expansion, Preconditioned Conjugate Gradient Method (PCG), and Conjugate Gradient Least Square Method (CGLS).

Introduction

Finding an unknown obstacle and its resistive properties is one of the crucial applications in inverse problem design and optimization. This encourages us to solve an inverse Cauchy problem governed by a modified Helmholtz equation to determine the temperature on the inner boundary of an annular region.

In this paper, we examine the inverse problem, which involves estimating the temperature u at the inner boundary of an annular domain from Cauchy data on the outer boundary (boundary temperature and heat flux), assuming that the steady state temperature u satisfies the modified Helmholtz equation governing the heat conduction in a fin.

$$\nabla^2 u - k^2 u = 0\,, \quad \Omega/D$$

When the domain Ω/D is a subset of R^2, Ω is an annular subset ,D is a defect which is also an anular subset of R^2.

Based on the knowledge of the boundary conditions[5] on the boundary ∂D of D and the knowledge of the Dirichlet temperature data $\frac{\partial u}{\partial n}$ and Neumann heat flux data on the outer half of the boundary $\partial\Omega$ of Ω , where $\frac{\partial u}{\partial n}$ is outward unit normal at [Lesnic & Bin-Mohsin, 2004,[20]]. These kinds of problems are ill-posed. In reality, a problem is well-posed in the sense of Hadamard if a unique, stable solution exists. If the solution does not satisfy one of these characteristics, the problem is ill-posed, and an inverse problem must be formulated to solve it. In contrast to direct problems, the inverse problem is typically known to be more challenging to solve.

When m increases, the error decreases up to the optimal m and then increases as it moves away from the optimal solution.

In addition, the inverse problems are unstable, Hadamard, 1923, [10], meaning that even a minor measurement error in the input data might result in a significant inaccuracy in the solution. Inverse problems have recently been considered in a number of scientific fields [18]. One of the inverse problem examples is the inverse Cauchy problem [25], [Hernandez [11]],[Lavrent'ev, 1986,[19], [21], and [Nachaoui et al., 2021[25]] are some references to this. In this type of problems, the boundary conditions (Dirichlet, Neumann) are only known for a portion of the boundary (the accessible portion), while the remaining portion of the boundary has no information, which makes it under-specified or inaccessible.

In order to avoid the ill-posedness of this kind of problem, a suitable algorithm must be selected for these problems. The Cauchy problem of the Helmholtz equation has been solved using a variety of techniques over the past 20 years. We will now briefly review a few of these techniques, including the truncation method [28], the conjugate gradient method (Marin et al., 2003,[24]), the meshless generalized finite difference method (Hua et al., 2017,[12]), the

Landweber method (Yang et al., 2017,[28]), and the fractional Tikhonov regularization method (Qian & Feng, 2017,[26]).

In fact, the quality of approximation is significantly impacted by the direct Helmholtz equation numerical solutions' dependency on the physical parameter$k$; for additional information, see [Ihlenburg & Babuska, 1995,[15]. [Berntsson et al., 2014, 2017, 2018,[5]], and [Qian & Feng, 2017,[26]] for some approaches that have been suggested to solve the Cauchy Helmholtz equation for some large parameters k. [Jourhmane & Nachaoui, 1996] suggested an alternating algorithm based on the relaxation of alternating algorithms. In [Berdawood et al., 2021[3]], the authors demonstrated that an efficient relaxed alternating approach proved the convergence for all values of wave number k in the case of the Helmholtz equation and accelerated the convergence in the case of the modified Helmholtz equation.

In order to approximate the solution of a Cauchy problem for a modified Helmholtz-type equation in a bounded domain surrounded by a smooth boundary, the goal of this work is to investigate an approach based on polynomial expansion. In this work, the meshless method suggested by Rasheed et al. [Rasheed et al. 2021,[27]] is used to approximate the temperature on the inaccessible inner boundary. This approach was well considered by Rasheed et al. in 2021,[27], to solve an inverse Cauchy problem and by Jameel et al. in 2022,[16], to solve a Cauchy problem Helmholtz equation.

The paper proceeds as follows. Section 2 presents basic definitions of the inverse Cauchy problem for the modified Helmholtz equation. Section 3 provides our proposed approximation method. In section 4, we solving two different examples numerically by using CGLS and PCG to solve the linear system.

Inverse Cauchy problem for the modified Helmholtz equation

Consider the domain with $\Omega/D \subset R^2$ where

Ω={(r,θ): 0≤ $r < 1, \quad 0 \le \theta \le 2\pi$}

D={(r,θ): 0≤ $r < \beta, \quad 0 \le \beta < 1, 0 \le \theta \le 2\pi$}

The domain $\Omega \subset \mathrm{R}^2$ has as boundary $\partial\Omega = \Gamma_{\mathbf{1}} \cup \Gamma_{\mathbf{2}}$ with

$\Gamma_1 = \{(r,\theta) \quad : r = \rho_e(\theta) \qquad 0 \le \theta \le 2\pi\}$ outer boundary

$\Gamma_2 = \{(r,\theta) \quad : r = \rho_i(\theta) \qquad 0 \le \theta \le 2\pi\}$ inner boundary

We consider the inverse Cauchy problem for the modified Helmholtz equation given in the following

$\Delta u(x,y) - k^2 u(x,y) = F \qquad \Omega/D$ (1)

$u(\rho,\theta) = h(\theta) \qquad on\ \partial_\Omega = \Gamma_1$ (2)

$\frac{\partial_\mu}{\partial_n}(\rho,\theta) = g(\theta) \qquad on\ \partial_\Omega = \Gamma_1$ (3)

When the domain Ω/D is a subset of R^2, Ω is an annular subset ,D is a defect which is also an anular subset of R^2.

Note On the portion of the domain boundary that is available, Cauchy data $u(x,y)$ and $\partial_n u(x,y)$ are provided. where the functions h(θ) and g(θ) are given

The part Γ_2 of the boundary is underdetermined, component Γ_1 is (overdetermined) and has two boundary conditions (no boundary condition is specified). To establish the temperature u on the inner boundary with an underdetermined value of Γ_2 , the inverse problem for the modified Helmholtz equation is formulated.

Remembering that Rasheed et al. (2021) and Liu and Kuo (2016) provide the following expressions for the normal derivative of u, denoted by $\partial_n u$:

$\partial_n u(\rho,\theta) = \eta(\theta)[\frac{\partial u(\rho,\theta)}{\partial \rho} - \frac{\rho'}{\rho^2}\frac{\partial u(\rho,\theta)}{\partial \theta}]$ (4)

$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta)+[\rho'(\theta)]^2}}$ (5)

We may also express the normal derivative $\partial_n u(x,y)$ by ∂nu in terms of $\partial_x u$ and $\partial_y u$.

$\partial_n u = \eta(\theta)[\cos\cos(\theta) - \frac{\rho'}{\rho^2}\sin\sin(\theta)]\,\partial_x u + \eta(\theta)[\sin\sin(\theta) - \frac{\rho'}{\rho^2}\cos\cos(\theta)]\partial_y u$ (6)

Approximation of the solution by polynomial expansion

The solution $u(x,y)$ can be expressed as a polynomial expansion

$$u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} cij x^{i-j}\, y^{j-1} \qquad (7)$$

The coefficients $u(x,y)$ must be established in order to find cij. The total number of these coefficients is $n = \frac{m(m-1)}{2}$, while the aforementioned polynomial's highest order is $m-1$.

Equation (8) allows us to determine $\partial_x u(x,y), \partial_y u(x,y), and\ \Delta u$

$$\partial_x u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(i-j)x^{i-j-1}\, y^{j-1} \quad (8)$$

$$\partial_y u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}(j-1)x^{i-j}\, y^{j-2} \quad (9)$$

$$\Delta u(x,y) - k^2 u(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}[(i-j)(i-j-1)x^{i-j-2}\, y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} - k^2(x^{i-j}y^{j-1}) \quad (10)$$

Initially, the coefficients in (8) c_{ij} can be written as an n-dimensional vector c with the components $c_{k,k=1,\dots\dots,n}$. The coefficients c_{ij} are really rearranged by taking $i = 1, \dots . m$ and $j = 1, \dots, i$ into account, with each index ij corresponding to one index k by taking $k = \frac{i(i-1)}{2} + j$. The vector a^T with inner product can be used to express the phrase $u(x,y)$.

$$u(x,y) = [1\ x\ \ y\ \ x^2\ xy\ \ y^2\ \ x^3 x^2 y\ \ xy^2\ \ y^3 \dots y^{m-1}][\mathrm{c}_1 \mathrm{c}_2\ \dots \mathrm{c}_n\ ]^T = a^T c \quad (11)$$

When we substitute (8) and (9) into (6), we get an expression for $\partial_n u$. The normal derivative $\partial_n u$ can also be represented for each point on the accessible portion of the boundary Γ_1 as the inner product of a vector e with c, where the $l-th$ component of e is given by:

$$e_l = \eta(\theta)[(i-j)x^{i-j-1}y^{j-1}(\cos\theta) - \frac{\rho'}{\rho^2}\sin(\theta)) + (j-1)x^{i-j}y^{j-2}\,(\sin\theta) - \frac{\rho'}{\rho^2}\cos(\theta))] \quad (12)$$

Keeping the same coefficients i,j for those who used $c_{i\,j}$ to derive e_l for $l\ = 1,\dots,n$. Now, from (10) the term $\Delta u(x,y) - k^2 u(x,y)$ can be stated for each point in the domain as the inner product of a vector d with c, where the $l-th$ component, l=1,...,n, is given by

$$d_k = (i-j)(i-j-1)x^{i-j-2}y^{j-1} + (j-1)(j-2)x^{i-j}y^{j-3} - k^2(x^{i-j}y^{j-1}) \quad (13)$$

The boundary condition (2)–(3) is verified by selecting n1 points on boundary Γ_1, say $(xi, yj) = (\mathrm{r}\cos(\theta_i), \mathrm{r}\sin(\theta_i)), and\ i = 1,2, \dots \dots n_1$. We also select n_2 points in the domain Ω/D, say $(x_{j,}y_j)$ and $j = 1, \dots \dots, n_2$ (to meet the equation) (1),

As a result, we have the linear system $Ac = b$ (14)

The vector $\boldsymbol{b}$ is hence longer $(2n_1 + n_2) \times 1$ and $\boldsymbol{A}$ is $(2n_1 + n_2) \times \frac{m(m+1)}{2}$ matrix provided in each case by:

$$\boldsymbol{A} = [a_1 \dots a_{n1} e_1 .. e_{n1} d_1 .. d_{n2}]^T$$

$$\boldsymbol{b} = [h(\theta_1) \dots h(\theta_{n1}) g(\theta_1) \dots g(\theta_{n1}) \dots, F(\theta_{n1})]^T \tag{15}$$

Solving the linear system

In order to solve the linear system $Ac = b$, we employ the widely used Preconditioned Conjugate Gradient Method **(PCG)** and the Conjugate Gradient least square method (CGLS).

1. **Algorithms of the Preconditioned Conjugate Gradient method (PCG) and the Conjugate Gradient least square method (CGLS)**

Preconditioned Conjugate Gradient method will at first be described as an iterative method to solve a linear system of equations Ax=b , A ∈ $R^{n,n}$ symmetric and positive definite .

Preconditioned Conjugate Gradient method (PCG)	**Conjugate Gradient least square method (CGLS)**
Algorithm 1: Preconditioned Conjugate Gradient method (PCG) **1.** let $\alpha_{k+1} = r_k^T z_k/(p_{k+1}^T w)$ **2.** let $\boldsymbol{x_{k+1} = x_k + \alpha_{k+1} p_{k+1}}$ **3.** let $\boldsymbol{r_{k+1} = r_k - \alpha_{k+1} w}$ **4.** let $\boldsymbol{p_{k+1} = z_k + \beta_k p_k}$ **5.** let $\boldsymbol{\beta_k = r_k^T z_k}/(\boldsymbol{r_{k-1}^T z_{k-1}})$ **6.** let $\boldsymbol{k = k + 1}$ **7.** repeat the prior actions until convergence	**Algorithm 2: Conjugate Gradient least Square Method (CGLS)** **1. let** $\alpha_k = \frac{\|r_k\|_2^2}{(p_k^T A^T)(Ap_k)}$ 2. **let** $\boldsymbol{x_{k+1} = x_k + \alpha_k p_{k\nabla}}$ **3. let** $\boldsymbol{r_{k+1} = A^T s_{k+1}}$ **4.** $\boldsymbol{p_k = -r_k + \beta_{k-1} p_{k-1}}$ **5. let** $\boldsymbol{\beta_k = \frac{\|r_{k+1}\|_2^2}{\|r_k\|_2^2}}$ **6. let** $\boldsymbol{k = k + 1}$ **7. repeat the prior actions until convergence**

2. **Stopping criterion and Initial guess**

The condition under which the algorithm can stop is crucial for any numerical approach, so we selected the following halting criteria:

$$\|r_i\| < Tol \tag{16}$$

$$\frac{\|r_i\|}{\|b\|} < Tol \tag{17}$$

Polynomials exact solution.

Here, we examine a Cauchy problem with a modified Helmholtz equation and an exact polynomial solution.

Example (1)

We consider the Cauchy problem for a modified Helmholtz equation with exact solution $u(x,y) = 6x^2y^2 - x^4 - y^4$, defined in an annular domain with constant radius $\rho_e = 1\ and\ \beta = 0.5$. This problem is over –specified on the following cases of the outer boundary

Case 1: $\Gamma_1 = \{(x,y): x^2 + y^2 = 0.5\}$

Case 2: $\Gamma_2 = \{(r,\theta)$:r(θ) =0.6+0.125cos(3θ)}

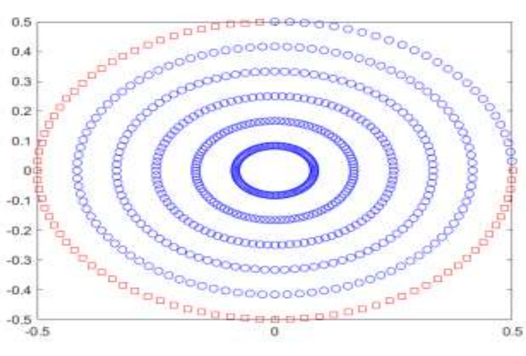
for which we have the following Cauchy data $h = 6x^2y^2 - x^4 - y^4$, $g = (12xy^2 - 4x^3)\cos(\theta) + (12x^2y - 3y^2)\sin(\theta)$. We study different cases for a different physical parameter k. For the numerical computations, we take $n_1 = 100, n_r = 5$, $n_2 = 500$ and so we take m =2,……,10 .we compare the results obtained by using the both algorithms CGLS and PCG with $tol = 10^{-10}$.

Tables 1 and 2 show the results for the **first case** of the boundary for the cases $k = \sqrt{15}$, $k = \sqrt{52}$.

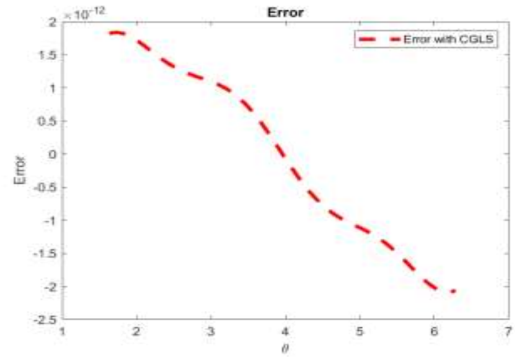
Table 1: $k = \sqrt{15}$

m	**No. of iteration**	Error BY CGLS	**No. of iteration**	Error BY PCG
2	3	1.00007397E+00	3	1.00007397E+00
3	7	1.00324586E+00	7	1.00324586E+00
4	14	1.04196701E+00	14	1.04196701E+00
5	**28**	**2.95804831E-11**	**29**	**2.25815537E-12**
6	58	6.08808211E-12	60	3.45335585E-10
7	131	3.44553652E-10	147	1.41125096E-10
8	296	4.64675328E-09	361	6.26365606E-08
9	694	9.08278950E-06	983	8.78152914E-06
10	1647	2.85148476E-03	3497	2.10013063E-05

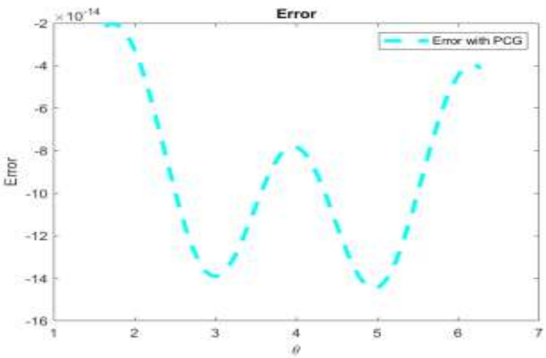
In table.1, we note that the best accuracy is obtained for $m = 5$,for both CGLS and PCG.



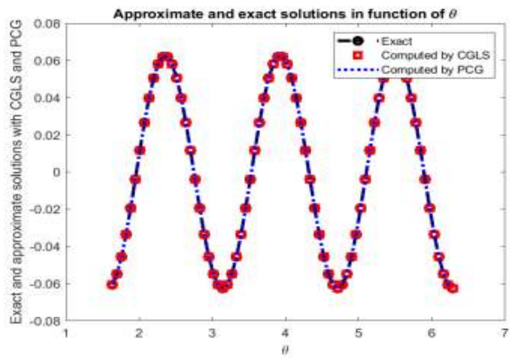
a)The domain of the problem



b) Error with CGLS



c) Error with PCG



d) Exact and Approximate solutions

Figure 1: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (1) with **:** $k = \sqrt{15}$ for (case 1).

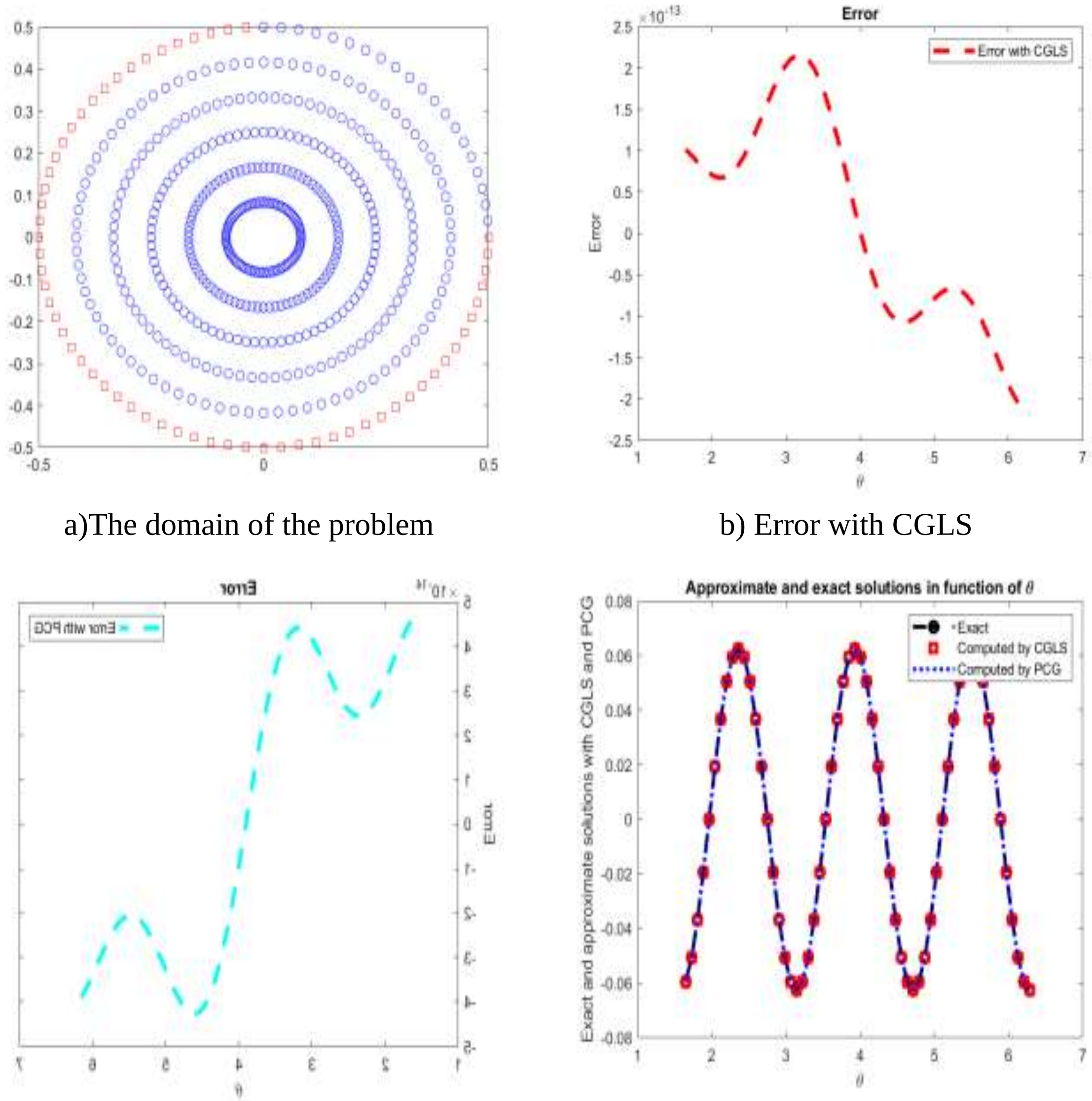
In the following table the results corresponding the case k=$\sqrt{52}$:

Table 2: k=$\sqrt{52}$

m	**No. of** iteration	Error BY CGLS	**No. of** iteration	Error BY PCG
2	3	1.00001827E+00	3	1.00001827E+00
3	7	1.00001827E+00	6	1.00001827E+00
4	13	1.00752609E+00	13	1.00752609E+00
5	25	1.42355190E-13	25	8.19791073E-14
6	53	1.64180877E-10	54	3.49139718E-13
7	110	9.22068630E-13	111	2.84713698E-11
8	244	5.55458255E-10	240	1.12013124E-08
9	573	5.43878795E-08	681	5.14157750E-09
10	1225	1.35007689E-04	1574	1.28703861E-06

For table 2, we note the same remark as table 1. In the following the figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.

In the following figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.



a)The domain of the problem

b) Error with CGLS

c) Error with PCG

d) Exact and Approximate solutions

Figure 2: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (1) with **:** $k = \sqrt{52}$ for (case 1).

Tables 3 and 4 show the results for **<u>case 2</u>** of the boundary for the k =$\sqrt{52}$, k =$\sqrt{100}$.

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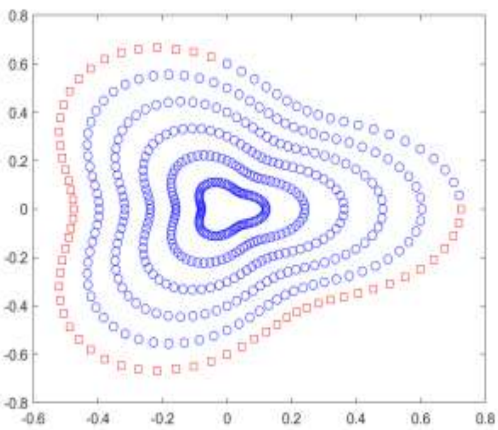
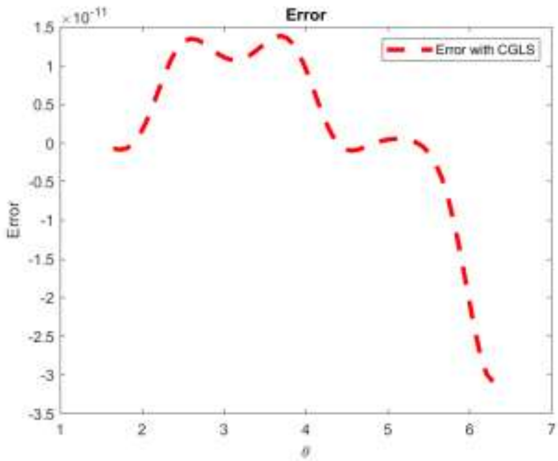
Table 3: $k = \sqrt{52}$

Error BY PCG	**No. of iteration**	Error BY CGLS	**No. of iteration**	m
2	3	9.39589272E-01	3	9.39589272E-01
3	7	9.38570919E-01	6	9.38570919E-01
4	14	9.31686911E-01	13	9.31686911E-01
5	**26**	**8.74862082E-11**	**26**	**1.02796244E-10**
6	48	1.60024323E-10	49	1.75617533E-09
7	100	1.84545729E-09	102	2.94427577E-09
8	212	4.49414480E-10	213	3.50778640E-09
9	487	1.04670877E-08	509	7.15219700E-09
10	907	1.09639801E-04	1006	1.09259481E-04

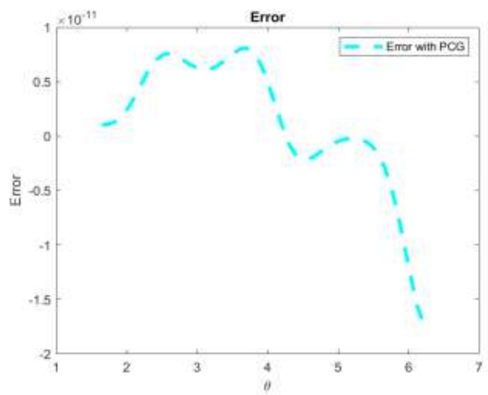
In table.3, we note that the best accuracy is obtained for $m = 5$,for both CGLS and PCG. In the following the figures for this case:



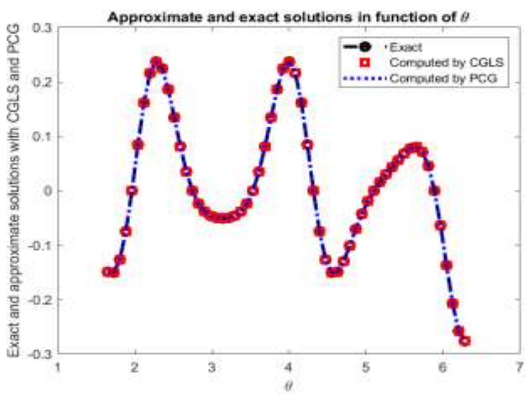
a)The domain of the problem



b) Error with CGLS



c) Error with PCG



d) Exact and Approximate solutions

Figure 3: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (1) with **:** $k = \sqrt{52}$ for (case 2).

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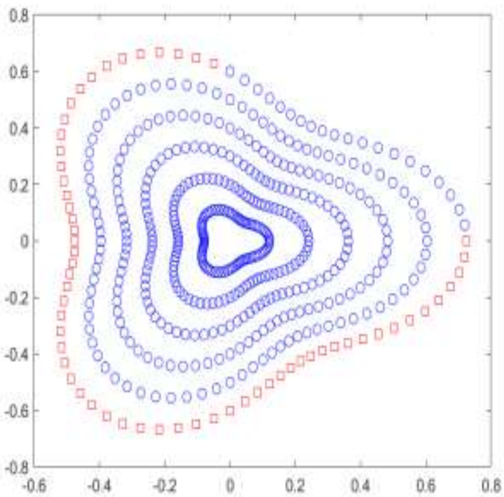
Now, we present the table and the figures for k=$\sqrt{100}$:

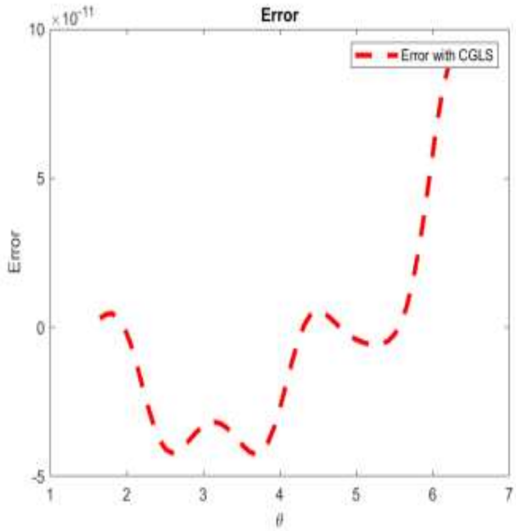
Table 4: k=$\sqrt{100}$

m	No. of iteration	Error BY CGLS	No. of iteration	Error BY PCG
2	3	9.41288071E-01	3	9.41288071E-01
3	7	9.41484415E-01	6	9.41484415E-01
4	14	8.19169114E-01	14	8.19169114E-01
5	**26**	**2.64346704E-10**	**27**	**1.90613455E-13**
6	55	3.23110272E-10	56	3.86034711E-12
7	110	4.10298918E-09	107	1.19666373E-08
8	232	7.07434255E-09	227	3.03141158E-09
9	522	2.21902176E-05	491	2.19547762E-05
10	737	1. 32608351E-03	727	1.32643012E-03

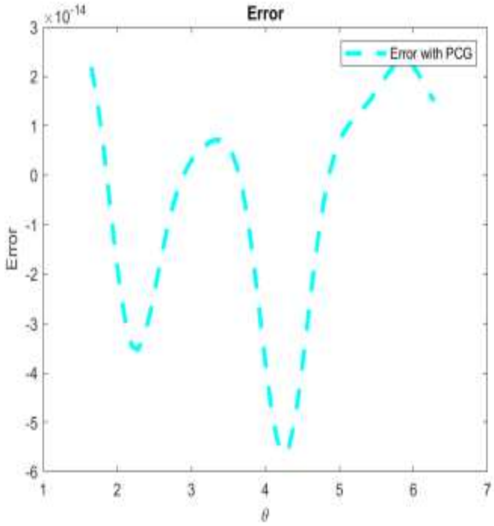
For table 4, we note the same remark as table 3. In the following the figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.



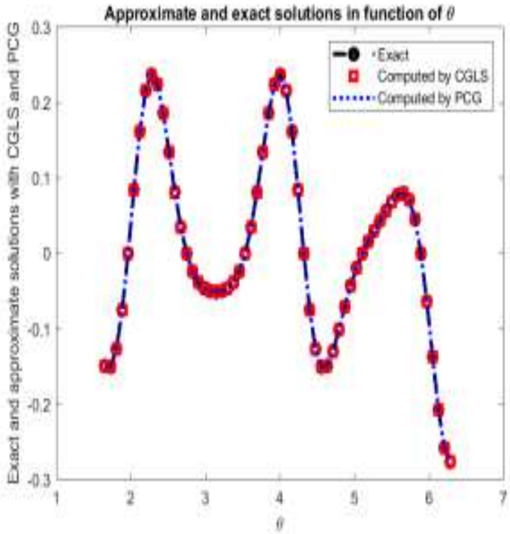
a)The domain of the problem



b) Error with CGLS



c) Error with PCG



d) Exact and Approximate solutions

Figure 4: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (1) with **:** $k = \sqrt{100}$ for (case 2).

We note that the tables (1,2,3,4) and the figures (1,2,3,4) show that the approximate solutions for the different case of the domain are obtained with high accuracy for the polynomial exact solution.

Non-Polynomial exact solution

Here, we examine a Cauchy problem with a non-polynomial exact solution arising from a modified Helmholtz equation.

Example (2): The Cauchy issue for a modified Helmholtz equation with an exact solution is taken into consideration $u(x) = exp(-x^2)$ defined in an annular domain with the constant radius $p_e = 1$and $\beta = 0.5$. This problem is over –specified on the following cases of the outer boundary

Case 1: $\Gamma_1 = \{(\mathrm{x}, \mathrm{y}): \mathrm{x}^2 + \mathrm{y}^2 = 0.5\}$

Case 2: $\Gamma_2 = \{(\mathrm{r}, \theta)$:r(θ) =0.6+0.125cos(3θ)}

we have the following Cauchy data $h = \exp(-x^2)$, $g = -2x\exp(-x^2)\ \cos(\theta)$. we study different cases for a different physical parameter k . For the numerical computations, we take

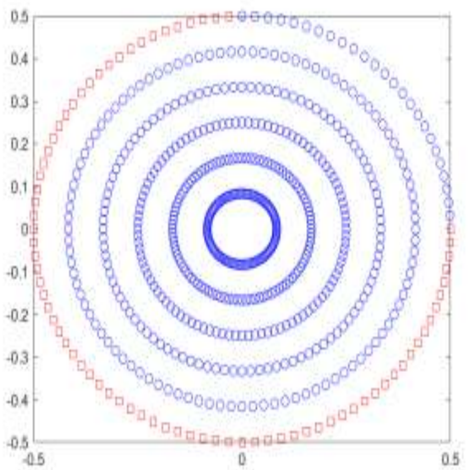
$n_1 = 100\,, n_r = 5\,, and\ so\ n_2 = 500$ and we take m = 2,3,4,………,15 we compare the results obtained by using the both algorithms CGLS and PCG with $tol = 10^{-12}$

Tables 9 and 10 show the results for the **first case** of the boundary for the cases k =$\sqrt{15}$, k =$\sqrt{25.5}$.

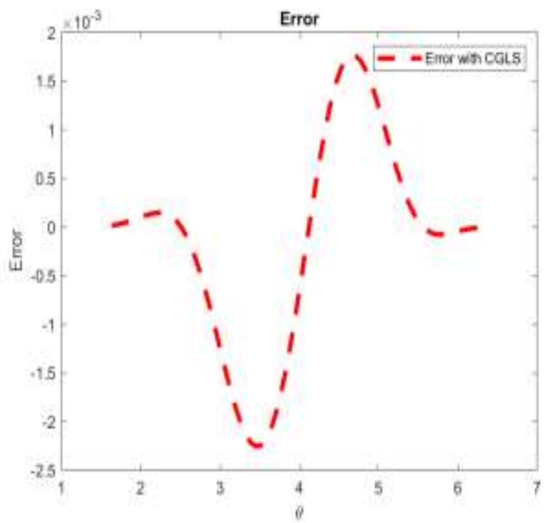
Table 5:k =$\sqrt{15}$

m	No. of iteration	ERROR BY CGLS	No. of iteration	ERROR BY PCG
2	3	2.40310943E-01	3	2.40310943E-01
3	7	1.00911166E-01	7	1.00911166E-01
4	13	1.30569379E-01	13	1.30569379E-01
5	26	4.87765441E-03	28	4.87765441E-03
6	57	9.01933329E-02	57	9.01933306E-02
7	120	1.44893838E-03	135	1.44877010E-03
8	204	7.58319767E-03	283	7.63147619E-03
9	**284**	**1.22165895E-03**	**337**	**1.22131928E-03**
10	341	1.22446828E-03	447	1.22422863E-03
11	281	1.76430500E-03	320	1.76414944E-03
12	259	1.76654578E-03	322	1.76643371E-03
13	360	1.39214603E-03	457	1.39128816E-03
14	364	1.39264445E-03	315	1.77214197E-03
15	309	1.62019373E-03	327	1.77287526E-03

In table.5, we note that the best accuracy is obtained for $m = 9$,for both CGLS and PCG.



a)The domain of the problem



b) Error with CGLS

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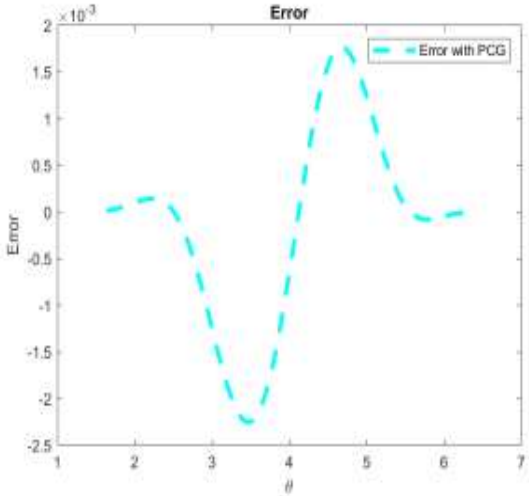
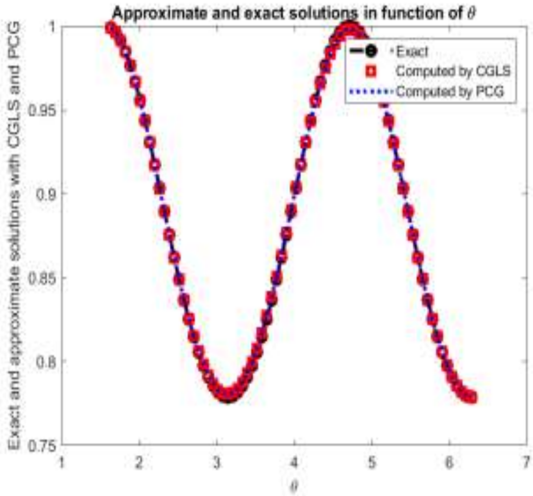




c) Error with PCG d) Exact and Approximate solutions

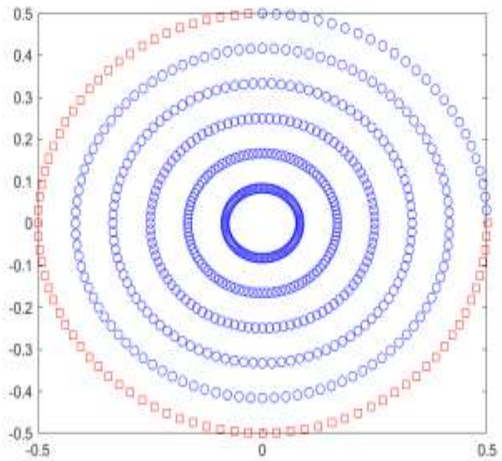
Figure 5: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (2) with **:** $k = \sqrt{15}$ for (case 1).

Table 6: $k = \sqrt{25.5}$

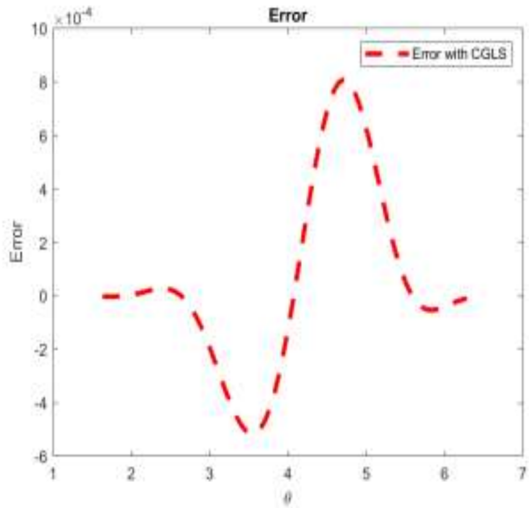
m	No. of iteration	Error BY CGLS	No. of iteration	Error BY PCG
2	3	1.89728008E-01	3	1.89728008E-01
3	7	5.27236034E-02	7	5.27236034E-02
4	14	5.85511176E-02	13	5.85511176E-02
5	27	5.64799730E-03	27	5.64799730E-03
6	57	3.61758072E-02	57	3.61758073E-02
7	121	5.12477470E-04	125	5.12476647E-04
8	218	8.92226699E-03	246	8.92228208E-03
9	**206**	**4.00790493E-04**	**220**	**4.00872462E-04**
10	220	4.26889835E-04	247	4.26884065E-04
11	266	4.91682224E-04	358	4.56434414E-04
12	296	4.57337142E-04	354	4.57312958E-04
13	255	5.02553139E-04	294	5.02525230E-04
14	257	5.02798818E-04	292	5.02773686E-04
15	289	4.56410238E-04	290	5.02733660E-04

For table 6, we note the same remark as table 5. In the following figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.

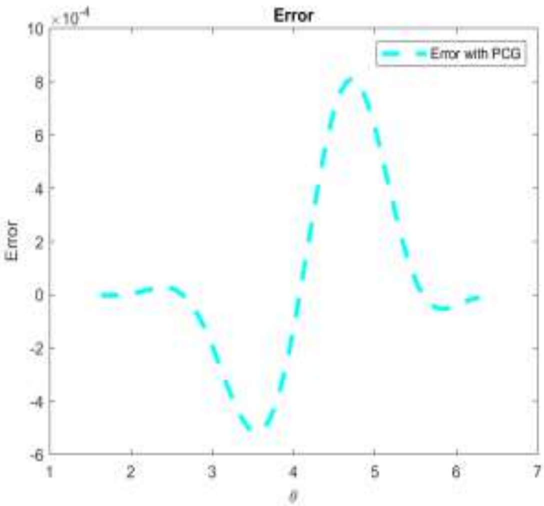
In the following figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.



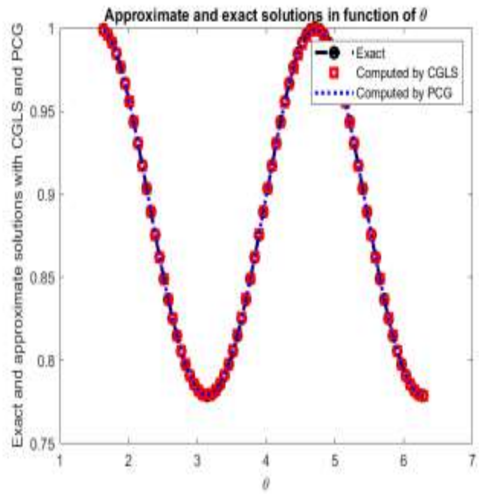
a)The domain of the problem



b) Error with CGLS



c) Error with PCG



d) Exact and Approximate solutions

Figure 6: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (2) with **:** $k = \sqrt{25.5}$ for (case 1).

Tables 7 and 18 show the results for **case 2** of the boundary for the cases $k = \sqrt{52}$, $k = \sqrt{100}$.

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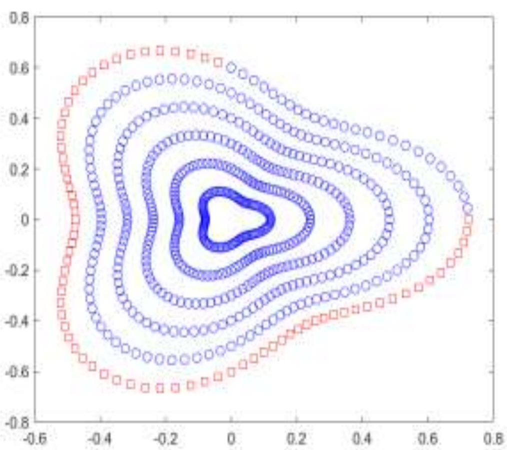
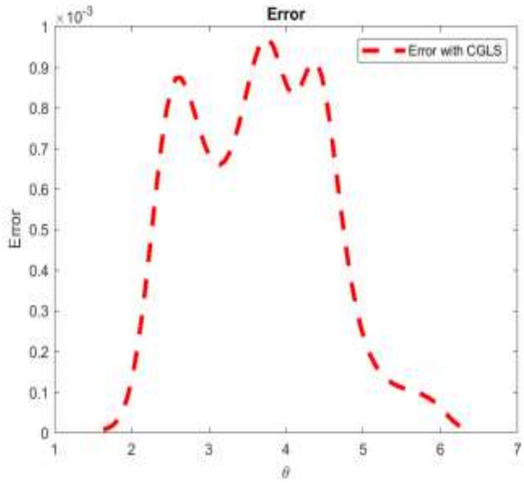
Table 7: k=$\sqrt{52}$

m	No. of iteration	Error BY CGLS	No. of iteration	Error BY PCG
2	3	2.15970724E-01	3	2.15970724E-01
3	7	3.96251795E-02	7	3.96251795E-02
4	14	4.37382993E-02	13	4.37382993E-02
5	26	7.75588663E-03	26	7.75588662E-03
6	54	1.44125730E-02	51	1.44125730E-02
7	113	2.43931965E-03	108	2.43931965E-03
8	225	5.50117327E-03	223	5.50117162E-03
9	325	9.12283326E-04	340	9.12337376E-04
10	398	1.44548332E-03	464	1.44595826E-03
11	350	9.05008495E-04	389	8.82823297E-04
12	**400**	**7.09539684E-04**	**443**	**7.09208938E-04**
13	396	7.61892186E-04	395	7.79435433E-04
14	392	8.15189756E-04	411	8.15160645E-04
15	395	8.12714219E-04	403	8.12738599E-04

In table. 7, we note that the best accuracy is obtained for $m = 12$,for both CGLS and PCG.



a)The domain of the problem



b) Error with CGLS

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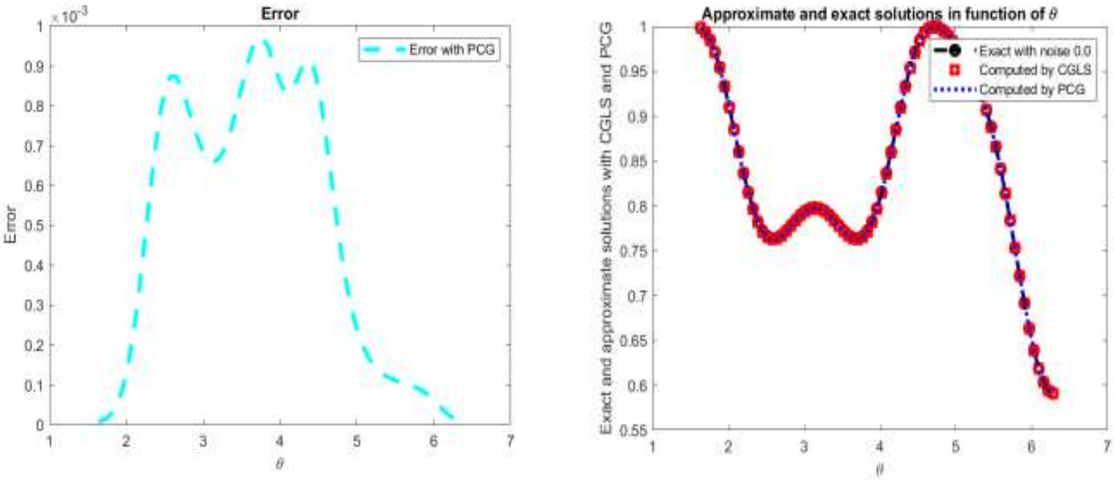


c) Error with PCG d) Exact and Approximate solutions

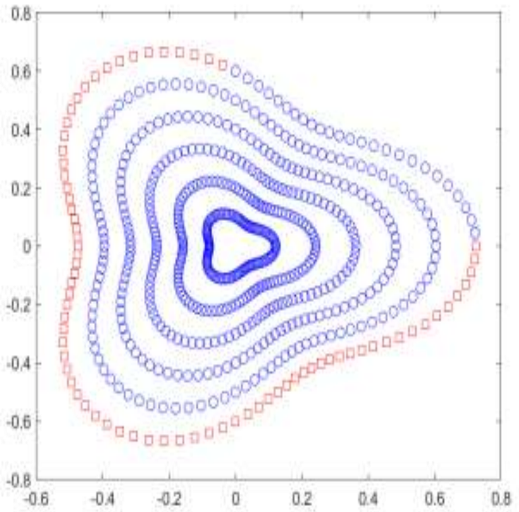
Figure 7: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (2) with **:** $k = \sqrt{52}$ for (case 2).

Table 8: k=$\sqrt{100}$

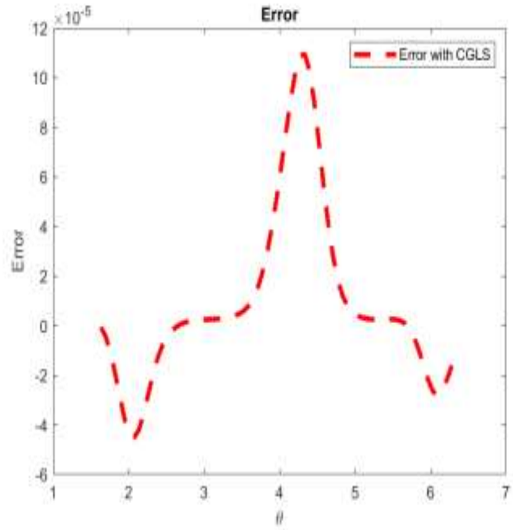
m	No. of iteration	Error by CGLS	No. of iteration	Error by PCG
2	3	2.00972037E-01	3	2.00972037E-01
3	7	2.75673714E-02	7	2.75673714E-02
4	13	2.87995425E-02	13	2.87995425E-02
5	26	4.40271831E-03	26	4.40271830E-03
6	55	5.19194429E-03	51	5.19194301E-03
7	112	5.22628870E-04	107	5.22623594E-04
8	235	1.32910153E-03	226	1.32909980E-03
9	292	1.79000941E-04	287	1.78948136E-04
10	414	2.13705172E-04	408	2.13566766E-04
11	361	1.85345408E-04	345	1.85398457E-04
12	**426**	**4.38836717E-05**	**302**	**9.44224662E-05**
13	427	6.52202274E-05	416	6.52283017E-05
14	418	6.06519942E-05	414	6.06716409E-05
15	426	5.35099023E-05	416	5.35081220E-05

For table 8, we note the same remark as table 7. In the following the figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.

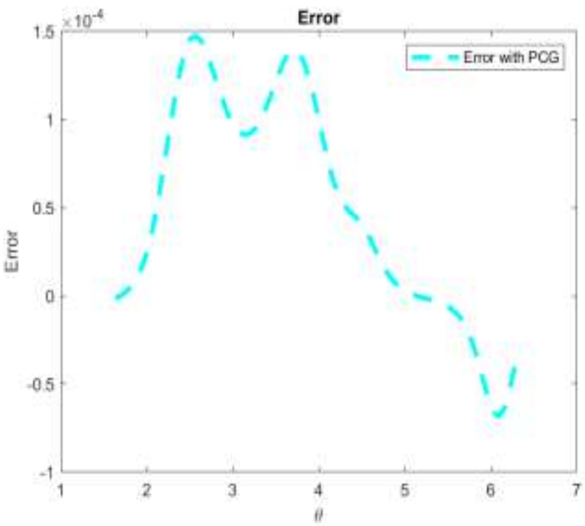
In the following the figures in which we present, the domain, a comparison between the exact solution and the approximate solution with CGLS and PCG and the error for these two methods.



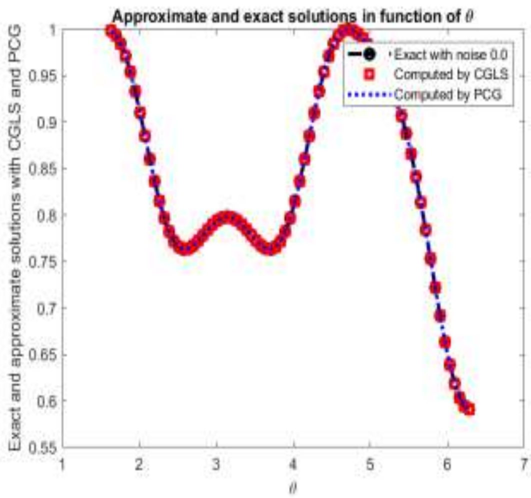
a)The domain of the problem



b) Error with CGLS



c) Error with PCG



d) Exact and Approximate solutions

Figure 8: The domain, the errors with CGLS and PCG and a comparision between the exact and approximate solutions for Example (2) with **:** $k = \sqrt{100}$ for (case 2).

We note that the tables (5,6,7,8) and the figures (5,6,7,8) show that the approximate solutions for the different case of the domain are obtained with high accuracy for the non-polynomial exact solution.

Stability and effect of a noise

The inverse problem is a type of issue caused by the collected (measured) data, and as these data may contain errors as a result of measurement mistake. The impact of a data noise on the approximation of the answer must therefore be studied. For this, we apply noise using the following form to the Cauchy data:

$$h(\theta) = u_{ex}(\rho, \theta) + \sigma * rand$$

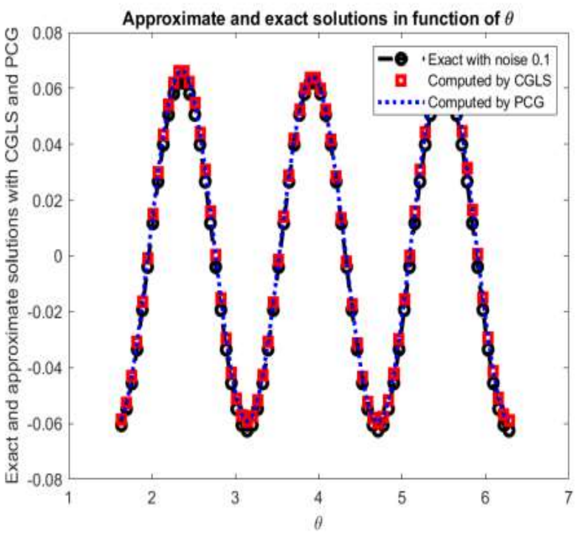
In general, in practical life matters, readings are taken using measuring instruments, and sometimes errors occur in the reading, so data is given with noise, an increase or decrease from the exact value of the measurement. Therefore, stability is tested by applying noise (an increase or decrease) randomly to the given data. (on the function or on the derivative).

For some measurement error deviation,$\sigma = 0.1, 0.05, 0.01, 0.001$ and for a Gaussian random error rand. We study the perturbation of Cauchy data by a noise for example 1, for a physical parameter $\sqrt{15}$,$n_1 = 100$, $n_2 = 500$ with $Tol = 10^{-10}$ and $\Gamma_1 = \{(x, y): x^2 + y^2 = 0.5\}$.

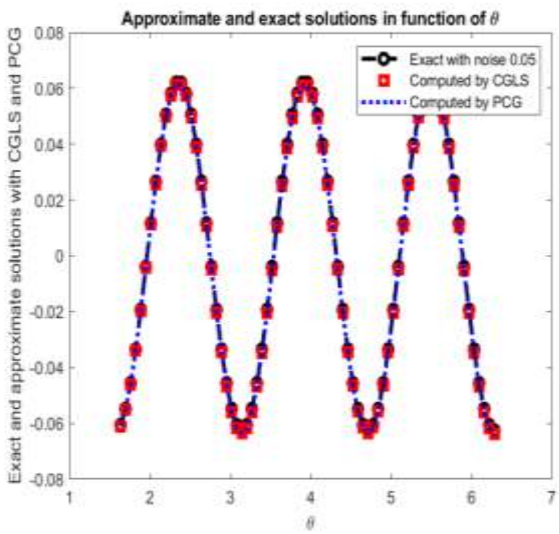
Table 9: Error and number of iteration for different level of noise for example (1)

σ	No. of Iteration for CGLS	Error with CGLS	No. of Iteration for PCG	Error with PCG
Without noise	**28**	**2.95804831E-11**	**29**	**2.25815537E-12**
0.1	27	5.18931534E-02	28	5.18931534E-02
0.05	27	8.22181676E-03	27	8.22181674E-03
0.01	28	2.65948335E-03	29	2.65948335E-03
0.001	27	1.11790842E-03	27	1.11790843E-03

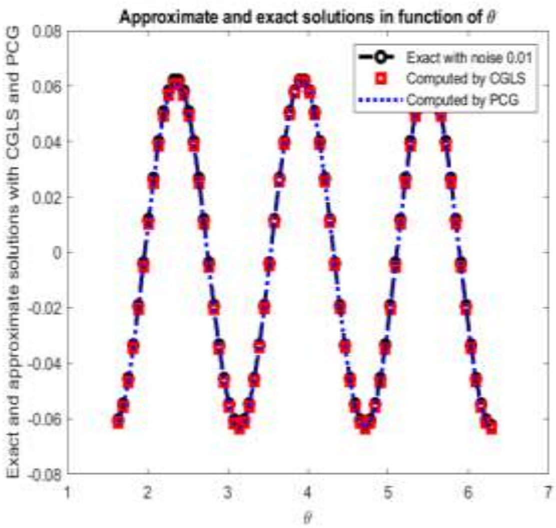
Now we present the figures show a comparison of the exact and approximate solution with the different noise level:



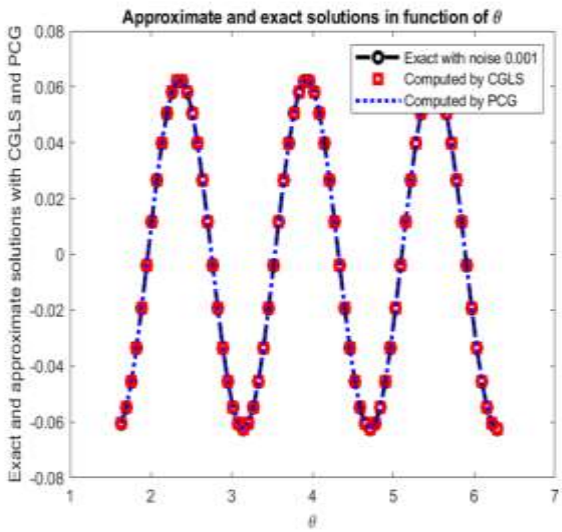
Noise parameter $\sigma = 0.1$



Noise parameter $\sigma = 0.05$



Noise parameter $\sigma = 0.01$



Noise parameter $\sigma = 0.001$

Figure 9: All noise level for Example 1

We note that for the different level of noise, the obtained approximate solution still approach the exact one with a good accuracy.

Conclusion

In order to find some unknown data on a portion of the boundary from supplied data on a different accessible portion, we solve the inverse Cauchy problem of the modified Helmholtz equation. The expression of the solution as a polynomial expansion, implies constructing a linear system, is used to transform the inverse Cauchy problem into a direct problem, which is then solved by (PCG) and (CGLS). By resolving some examples and contrasting the precision of (PCG) and (CGLS), the proposed method is confirmed to be effective in overcoming the ill-posed-ness of the inverse Cauchy Problem. Applying a noise to the Cauchy data allows for the investigation of the method's stability.

References

1. F. Aboud, A. Nachaoui, M. Nachaoui, On the approximation of a Cauchy problem in a non-homogeneous medium, J. Phs., Conf.Ser.,1743, (2021), DOI(10.1088/1742-6596/1743/1/012003)
2. F. Aboud, I. Th. Jameel, A. F. Hasan, B. Kh. Mostafa, A. Nachaoui, Polynomial approximation of an inverse Cauchy problem for Helmholtz type equations. Advanced Mathematical Models & Application, 7(3), 2022, 306-322(2022)
3. K. A. Berdawood, A. Nachaoui, M. Nachaoui, F. Aboud, An effective relaxed alternating procedure for Cauchy problem connected with Helmholtz equation, Numer Methods partial Differential Eq., 1-27(2021), DOI(https://doi.org/10.1002/num.22793)
4. K. A. Berdawood, A. Nachaoui, R. Saeed, M. Nachaoui, F. Aboud, An efficient D − N alternating algorithm for solving an inverse problem for Helmholtz equation, Discrete & Continuous Dynamical Systems-S, American Institute of Mathematical Sciences, 15(1), (2021), DOI(https://doi.org/10.3934/dcdss.2021013)
5. F. Berntsson, V. Kozlov, L. Mpinganzima, B. O. Turesson, Robin-Dirichlet algorithms for the Cauchy problem for the Helmholtz equation, Inverse Probl. Sci. Eng., 26, 1062–1078(2018), DOI(https://doi.org/10.1080/17415977.2017.1380639)
6. F. Berntsson, V.A. Kozlov, L. Mpinganzima, B.O. Turesson, Iterative Tikhonov regularization for the Cauchy problem for the Helmholtz equation, Comput. Math., Appl.,73(1), 163-172(2017), DOI(https://doi.org/10.1016/j.camwa.2016.11.004)
7. F. Berntsson, V. A. Kozlov, L. Mpinganzima, B.O. Turesson, An alternating iterative procedure for the Cauchy problem for the Helmholtz equation, Inverse Probl. Sci.Eng., 22, 45-62(2014), DOI(https://doi.org/10.1080/17415977.2013.827181)

8. F. Hasan, D. Jasim, I. T. Jameel, A. F. Hasan, B. K. Mostafa, F. Aboud, Polynomial approximation of an inverse Cauchy Problem for modified Helmholtz Equations, Academic Science Journal, 1(2), 153-170,(2022),DOI(https://dx.doi.org/10.24237/ASJ.01.02.650D)
9. F. Hasan, solving an inverse Cauchy Problem for modified Helmholtz Equations.Related to Boundary Heat Transfer, MSC thesis at Department of Mathematics, College of Science, University of Diyala, (2022)
10. J. Hadamard, Lectures on Cauchy problem in lin-ear partial differential equation, (Dover Publication, New York, 1953)
11. E. Hernandez-Montero, A. Fraguela-Collar, J. Henry, An optimal quasi solution for the Cauchy problem for Laplace equation in the framework of inverse ECG, Math. Model. Nat. Phenom., 14(2), (2019), DOI(https://doi.org/10.1051/mmnp/2018062)
12. Q. Hua, Y. Gu, W. Qu, W. Chen, C. Zhang, A meshless generalized finite difference method for inverse Cauchy problems associated with three-dimensional inhomogeneous Helmholtz-type equations, Eng. Anal. Bound. Elem., 82, 162–171(2017), DOI(https://doi.org/10.1016/j.enganabound.2017.06.005)
13. C. H. Huang, W. C. Chen, A three-dimensional inverse forced convection problem in estimating surface heat flux by conjugate gradient method, International Journal of Heat and Mass Transfer, 43(17), 3171-3181(2000), DOI(https://doi.org/10.1016/S0017-9310(99)00330-0)
14. F. Ihlenburg, I. Babuska, Finite element solution of the Helmholtz equation with high wave number part ii: The hp version of the fem, SIAM J. Numer. Anal., 34, 315–358(1997), DOI(https://doi.org/10.1016/0898-1221(95)00144-N)
15. F. Ihlenburg, I. Babuska, Finite element solution of the Helmholtz equation with high wave number part i: The h-version of the fem, Comput. Math. Appl., 30, 9–37(1995),DOI(https://doi.org/10.1016/0898-1221(95)00144-N)
16. I. T. Jameel, A. F. Hasan, D. Jasim, A. Nachaoui, F. ABOUD, Solving Inverse Cauchy Problems of the Helmholtz Equation Using Conjugate Gradient Based-Method: BICG,

BICGSTAB And PCG, Academic Science Journal, 1(4), 13-31(2023), DOI(https://doi.org/10.24237/ASJ.01.04.652D)

17. V. A. Kozlov, V. G. Maz'ya, A. V. Fomin, An iterative method for solving the Cauchy problem for elliptic equations, Zh. Vychisl. Mat. I Mat. Fiz., 31, 64–74(1991)
18. S. Kubo, Inverse problem related to the mechanics and fracture of solids and structures JSME Int J ,31,157-166,(1988), DOI(https://doi.org/10.1299/jsmea1988.31.2_157)
19. M. M. Lavrent'ev, V. G. Romanov, S. P. Shishatskiı, Ill-posed problems of mathematical physics and analysis, volume 64 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI. Translated from the Russian by J. R. Schulenberger, Translation edited by Lev J. Leifman, (1986)
20. D. Lesnic, B. Bin-Mohsin, Inverse shape and surface heat transfer coefficient identification. Journal of Computational and Applied Mathematics, 236(7), 1876-1891(2012), DOI(https://doi.org/10.1016/j.cam.2011.10.020)
21. C. S. Liu, F. Wang, A meshless method for solving the nonlinear inverse Cauchy problem of elliptic type equation in a doubly-connected domain, Comput. Math.Appl.,76, 1837–1852(2018), DOI(https://doi.org/10.1016/j.camwa.2018.07.032)
22. C. S. Liu, C. L. Kuo, A multiple-scale Pascal polynomial triangle solving elliptic equations and inverse Cauchy problems, Engineering Analysis with Boundary Elements, 62, 35-43(2016), DOI(https://doi.org/10.1016/j.enganabound.2015.09.003)
23. L, Marin, Numerical boundary identification for Helmholtz –type equation, J. comput .Mech., 39, 25-40(2006), DOI(https://doi.org/10.1007/s00466-005-0006-9
24. L. Marin, L. Elliott, P. J. Heggs, D. Lesnic, X. Wen, Conjugate gradient-boundary element solution to the Cauchy problem for Helmholtz-type equations, Computational Mechanics, 31. 367-377(2003), DOI(https://doi.org/10.1007/s00466-003-0439-y)
25. A. Nachaoui, M. Nachaoui, A. Chakib, M. A. Hilal, Some novel numerical techniques for an inverse Cauchy problem, Journal of Computational and Applied Mathematics, 381,113030(2021), DOI(https://doi.org/10.1016/j.cam.2020.113030)

26. Z. Qian, X. Feng, A fractional Tikhonov method for solving a Cauchy problem of Helmholtz equation ,Appl.Anal, 96 , 1656-1668(2017), DOI(https://doi.org/10.1080/00036811.2016.1254776)
27. S. M. Rasheed, A. Nachaoui, M. F. Hama, A. K. Jabbar, Regularized and preconditioned conjugate gradient like-methods for polynomial approximation of an inverse Cauchy problem, Advanced Mathematical Models & Applications, 6(2), 89-105(2021)
28. F. Yang, P. Zhang, X. X. Li, The truncation method for the Cauchy problem of the inhomogeneous Helmholtz equation, Appl. Anal., 98, 991-1004(2019), DOI(https://doi.org/10.1080/00036811.2017.1408080)