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Polynomial Approximation of a Nonlinear Inverse Cauchy Problem

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Abstract

In this paper, a class of nonlinear inverse boundary problem in the context of heat transfer is considered. We consider a class of nonlinear inverse boundary problems in the context of heat transfer. The problem involves determining the temperature distribution within a domain subject to a Cauchy boundary condition on a part of its boundary. We introduce a transformed variable, which allows us to reformulate the problem as a linear Cauchy problem followed by a series of nonlinear equations. We propose a polynomial expansion method to solve the linear Cauchy problem for the Laplace equation, and we employ the Newton method to solve the resulting nonlinear equations. Importantly, our approach does not rely on mesh-based discretization, allowing for parallel computation and preserving the mesh-free nature of the problem. We present numerical results obtained using our methodology and discuss the effectiveness of the proposed approach. The results show that the method provides a robust and efficient framework for solving nonlinear inverse boundary problems in heat transfer, with potential applications in various engineering and scientific fields.

Keywords: Inverse Cauchy problems, Polynomial expansion, Nonlinear, Preconditioned linear systems.

Introduction

Inverse boundary problems play a crucial role in many scientific and engineering applications [1-26], and particularly in heat transfer $[2, 7, 27 - 32]$. These problems involve the determination of unknown parameters or fields inside a domain based on measurements or observations made on the boundary. A such class of problems is the nonlinear inverse boundary problem for the Poisson equation, which aims to recover the temperature distribution within a domain [33-35].

The linear Cauchy problem for the Poisson equation has been extensively studied in the literature $[16, 18, 36 - 39]$. It involves finding a solution to the Poisson equation inside a domain, given values on a subset of the boundary. Various numerical methods have been developed to tackle this problem, such as the boundary element method [33, 34, 40-42], finite element method [18, 34, 43-45], and finite difference method [28,46]. These methods typically rely on mesh-based discretization techniques and have proven to be effective in solving the linear Cauchy problem.

On the other hand, the polynomial expansion method has gained significant attention in solving boundary value problems [47-51]. This approach seeks to approximate the solution using a polynomial expansion, where the coefficients are determined by satisfying the governing partial differential equation and the boundary conditions. The advantage of this method lies in its ability to handle nonlinearities and its mesh-free nature, allowing for parallel computation and efficient solutions to problems with complex geometries.

When dealing with nonlinear inverse boundary problems, such as the one considered in this paper, the resolution of the Cauchy problem is followed by solving a system of nonlinear scalar equations. The Newton method is a widely used technique to solve such nonlinear equations. It involves iteratively updating an initial estimate by approximating the solution through linearization of the equations. The convergence and efficiency of the Newton method make it a popular choice for solving nonlinear problems.

In this paper, we propose a novel approach to solve the nonlinear inverse boundary problem for the Poisson equation. We first reformulate the problem as a linear Cauchy problem, which we solve using a polynomial expansion method. The mesh-free nature of the polynomial expansion allows for parallel computation and efficient solutions. We then employ the Newton method to solve the sequence of resulting nonlinear scalar equations, providing accurate and robust solutions to the problem.

The rest of the paper is organized as follows. In Section 2, we present the methodology, detailing the polynomial expansion method and the Newton method for solving the linear Cauchy problem and the nonlinear equations, respectively. In Section 3, we provide numerical results to demonstrate the effectiveness of our approach. Finally, in Section 4, we draw conclusions and discuss potential future research directions.

The inverse problem and the Methodology

Consider the following class of nonlinear inverse boundary problem:

$$\begin{cases} -\nabla K(T)\nabla T = f & on\ \Omega \\ T|_{\Gamma_d} = f_d & on\ \Gamma_d \\ T|_{\Gamma_2} = f_2 & on\ \Gamma_2 \\ K(T)\partial_\nu T|_{\Gamma_2} = g_2 & on\ \Gamma_2 \\ K(T)\partial_\nu T|_{\Gamma_n} = g_n & on\ \Gamma_n \end{cases} \quad \text{...(1)}$$

where $\Omega \subset \mathbb{R}^2$ is a domain for which the boundary Γ is such that $\Gamma = \Gamma_1 \cup \Gamma_d \cup \Gamma_2 \cup \Gamma_n$.

We denote by ν the outward normal vector to $\partial\Omega$ and by ∂_ν the normal derivative operator. Suppose that $\Gamma_1 \neq \emptyset$ and that no boundary condition is specified there. We also assume that the real function K is non-negative.

Note that, in the modeling of inverse boundary problems in heat transfer, $K(T)$ is the conductivity coefficient, T is the temperature distribution within the system denoted Ω, the function *f (x)* denotes the source term and Γ_1 is the inaccessible boundary.

To illustrate our procedure, let us introduce the transformed variable ω that satisfies the relationship $\nabla\omega = K(T)\nabla T$. This implies that the governing equation in problem (1) becomes the Laplace equation, and the problem (1) is reduced to solving the following linear Cauchy problem:

$$\begin{cases} -\Delta\omega = f & in\ \Omega \\ \omega \quad = F(f_d) & on\ \Gamma_d \\ \omega \quad = F(f_2) & on\ \Gamma_2 \\ \partial_v\omega \ = g_2 & on\ \Gamma_2 \\ \partial_v\omega \ = g_n & on\ \Gamma_n \end{cases} \quad \text{.. (2)}$$

followed by a series of nonlinear equations:

$$F\big(T(X)\big) = \omega(X) \quad \forall\ X \in \Omega \quad \text{.. (3)}$$

where $\omega(X)$ is the value of the solution of (2) at a point X in $\bar{\Omega}$, and F denotes the transformation formula defined by:

$$F(T) = \int_0^T K(t)dt \quad \text{...(4)}$$

Note that (2) is the classical linear Cauchy problem for the Laplace equation. To obtain the solution of the initial problem (1), one has to solve (2) using any method dedicated to solving the linear Cauchy problem, and then solve the scalar nonlinear equation (3) by any numerical method for solving the nonlinear equations. It should be noted that the resolution of the equation (3) for an $X \in \Omega$ is independent of any discretization of the domain, and the resolution for different points from Ω can be done in a parallel way. The method is therefore a method without mesh. We will use a polynomial expansion method, which preserves this characteristic, to solve the problem of Cauchy (2), and we will use the Newton method to solve the nonlinear equations (3) for any $X \in \Omega$.

Approximation of the Solution using Polynomial Expansion

To approximate the solution $\omega(x,y)$ of the Cauchy problem (2), we employ the collocation technique with the polynomial expansion method. The key idea is to expand $\omega(x,y)$ as a polynomial in x and y, allowing us to determine the coefficients by solving a system of equations.

We express the solution $\omega(x,y)$ as follows:

$$\omega(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} x^{i-j} y^{j-1}, \quad \text{.......................................(5)}$$

where m is the polynomial degree. The coefficients c_{ij} are the unknowns that we need to determine. Their number is given by $n = \frac{m(m-1)}{2}$. To obtain these coefficients, we substitute the expansion (5) into the equations of the Cauchy problem (2).

Let $X_k = (x_k, y_k)$ be the k-th collocation node, where $k = 1,2,\ldots,K$. The collocation nodes can be chosen based on various methods, such as equidistant nodes. These collocation nodes have to be located on the boundary and in the interior of the domain Ω. We evaluate the equations of (2) at these collocation nodes, resulting in a rectangular system of linear equations of the form:

$$Ac = b \quad \text{..(6)}$$

where c is the column vector of coefficients c_{ij}, b is the column vector containing the values obtained from evaluating the equations at the collocation nodes, and A is the coefficient matrix.

Once the coefficients c are determined, we can reconstruct the approximate solution $\omega(x,y)$ by substituting them back into the expansion (5). The resulting polynomial approximation provides an estimate for the solution of the Cauchy problem (2).

By utilizing the collocation technique with the polynomial expansion method, we are able to obtain an efficient and accurate approximation for the solution $\omega(x,y)$, without relying on mesh-based discretization. The flexibility of choosing the collocation nodes and the polynomial degree m allows for adapting the method to different problem settings and achieving desired accuracy. When the data are noisy or when certain approximations are used for the calculations, the obtained system can be ill conditioned. We can then use the regularization and the preconditioning developed [52], in order to improve the quality of the solution.

In the next section, we present numerical results obtained using this approach and discuss the performance and accuracy of the polynomial expansion method in solving the Cauchy problem.

Numerical Results

In this section, we present the numerical results obtained using the polynomial expansion method with the collocation technique for solving the linear Cauchy problem (2). We also discuss the efficiency of the method and the use of Newton's method for resolving the nonlinear scalar equations.

We consider various test cases with different geometries and boundary conditions to assess the accuracy and convergence of the proposed method. For each case, we choose a suitable polynomial degree m and distribute the collocation nodes accordingly.

The first aspect we examine is the efficiency of the polynomial expansion as an approximation of the linear Cauchy problem. The results demonstrate that the method provides accurate solutions without the need for mesh generation. This eliminates the computational overhead associated with mesh-based methods and allows for efficient computations.

Furthermore, we utilize Newton's method to solve the nonlinear scalar equations arising from the polynomial expansion. Newton's method proves to be highly effective in finding the solutions, providing rapid convergence. The nonlinear equations are efficiently resolved, leading to accurate approximations of the solution.

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For all the numerical examples we present here, the domain defined by $\Omega = (0,1) \times (0,1)$ will be considered. The different components of the boundary $\Gamma$of this domain will be defined by: $\Gamma_2 = \{(x,y): y = 0\}$, $\Gamma_n = \{(x,y): x = 0\}, \Gamma_d = \{(x,y); x = 1\}$ and $\Gamma_1 = \{(x,y): y = 1\}$.

In the sequel, we will use the following abbreviations: **Reg** to say regularization, **Prec** designate precondition, **MP** means the number of products in the preconditioner, **TSkMP** denotes the two-sided k-multi-preconditioned system, **ND** designate normal derivative, **IterN** and **IterC** to indicate the number of iteration necessary for the convergence of Newton's and CGLS method respectively. All algorithms proposed in this paper were programmed using MATLAB.

Example (1):

For this first example, we consider the conductivity given by $K(T) = exp(T)$ and given data computed from the analytical solution $T(x,y) = ln(2 + x^2 - y^2)$. We take $tol = 10^{-12}$ as tolerance for linear system solver CGLS and the value $e = 10^{-12}$ was taken in the stopping criterion of Newton's method. Note that, for this example, the integrals in equation (2) can be computed exactly using analytic integration.

Table 1: Error for Example 1 by using exact integration.

m	ERROR
$m = 1$	2.81859193838558
$m = 2$	2.23975626768953
$m = 3$	4.10443728116111e − 13
$m = 4$	4.42122652660508e − 13
$m = 5$	3.07966518201704e − 12
$m = 6$	1.90873478677371e − 11
$m = 7$	1.11199338659765e − 10
$m = 8$	1.48429171122976e − 09
$m = 9$	2.36475411789684e − 06
$m = 10$	8.96262110980807e − 05

To illustrate the effectiveness of the methodology, we present several tables and figures showcasing the results.

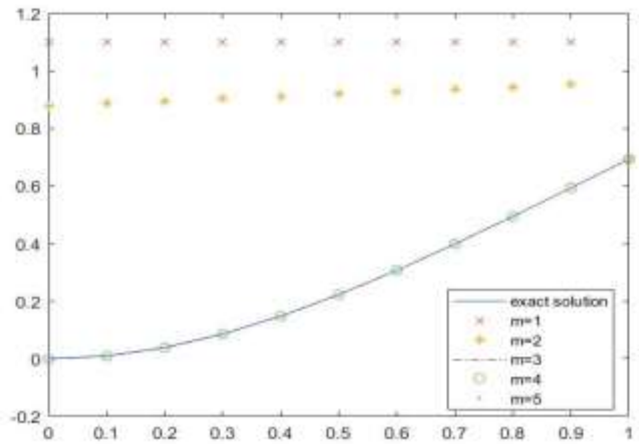

Figure 1: Solution to Example 1 with analytic integration for various m .

Table 1 and Figure 1 present the convergence behavior of the polynomial expansion method as the polynomial degree m increases. It shows the decreasing error norms with respect to m, indicating that higher polynomial degrees lead to more accurate solutions. However, there is a trad-off between accuracy and computational complexity, as higher degrees require more computational resources. We notice that the best result is obtained for $m = 3$ which is in perfect agreement with the results shown in [48,49], namely, that the best approximation obtained by the polynomial expansion when the data is calculated from a polynomial function of degree q is $m = q + 1$. This is the case here since for this example, the exact solution of the linear Cauchy problem is:

$$\omega(x, y) = \int_0^{T(x,y)} K(t)dt = 1 + x^2 - y^2$$

Which is a polynomial of degree 2.

To test the influence of the initial iterate, we fixed $m = 3$ and took different value of T_0 . The obtained results are in table 2.

Table 2: Results of error for Example 1 by using exact integration.

	ERROR	ERROR ON ND	ITERN	ITERC
$T_0 = 0$	4.104e-13	1.0228e-13	5	7
$T_0 = 5$	4.421e-13	1.0228e-13	11	7
$T_0 = -5$	4.227e-13	1.0228e-13	191	7

We see from Table 2 that the calculated solution as well as its normal derivative, obtained for various initial iterates are of excellent quality and they are of the same order. The error for the solution and its normal derivative is of the order of 10^{-13}. It is concluded that the accuracy obtained by the method is insensitive to the initial iteration. Only the number of iterations is modified, which is a characteristic of Newton's method.

Table 3: Error for Example 1 by using Numerical integration.

m	ERROR
$m = 1$	2.81859193838558
$m = 2$	1.90194989751648
$m = 3$	3.19724523644109e-06
$m = 4$	3.28780291911860e-06
$m = 5$	3.33384923627198e-06
$m = 6$	8.06413245008753e-06
$m = 7$	1.01867005724143e-05
$m = 8$	1.17742715193908e-05
$m = 9$	1.11093353994306e-05
$m = 10$	6.90340396632885e-04

As generally, integral in equation (2) cannot be calculated analytically, one must use a numerical integration. We present in Table 3 the results obtained by using the Trapezoidal Rule Formula. We observe a loss of precision, which is normal since the calculation of the solution uses non-exact results. But we also note that these results remain very satisfactory, the error for $m = 3$ is of the order of 10^{-6} without regularization without preconditioning and of 10^{-7} when the regularization is applied as shown in Table 4.

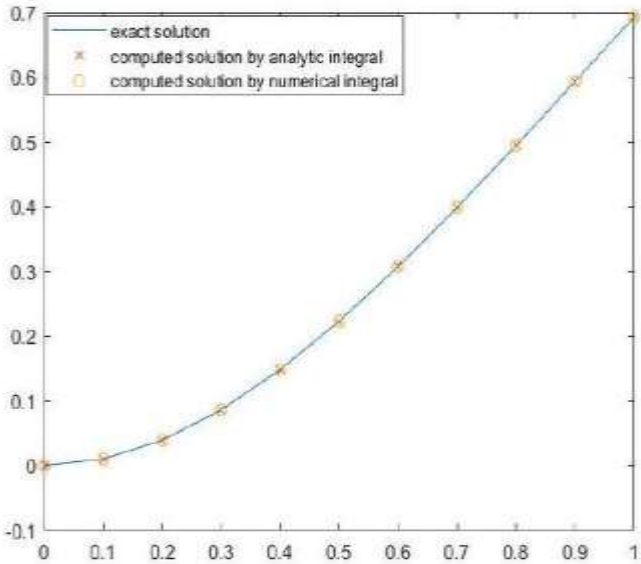

Figure 2: Exact and reconstructed solutions on Γ_1 in Example 1.

We observe from Figure 2 that the solution obtained with or without numerical integration are confused with the exact solution. This is why we will adopt numerical integration for the following results, which is more reasonable since in real life problems we can rarely use analytical integration.

Table 4: Results for Example 1 when $T_0 = 0$.

	Γ	MP	ERROR	ERROR ON ND	ITERN	ITERC	M
No Reg. And No Prec.	-	-	3.197e-06	9.695e-06	5	8	-
Reg. And No Prec.	-	-	4.367e-07	3.847e-06	5	8	1.00e-06
TSkMP	0.4	1	2.686e-07	3.117e-06	5	7	1.00e-08

For exact integration, we have already observed that the solution does not depend on the initial iterate, we then examine the case of numerical integration.

Table 5: Results for Example 1 when $T_0 = 5$.

	γ	MP	Error	Error on ND	IterN	IterC	μ
No Reg. And No Prec.	-	-	3.197e-06	9.695e-06	11	8	-
Reg. And No Prec.	-	-	4.367e-07	3.847e-06	11	8	1.00e-06
TSkMP	0.1	2	2.342e-07	3.357e-06	11	8	1.00e-16

Table 6: Results for Example 1 when $T_0 = -5$.

	γ	MP	Error	Error on ND	IterN	IterC	μ
No Reg. And No Prec.	-	-	3.197e-06	9.695e-06	87	8	-
Reg. And No Prec..	-	-	4.367e-07	3.847e-06	87	8	1.00e-06
TSkMP	0.1	2	2.342e-07	3.357e-06	87	8	1.00e-06

We observe from Tables 4-6 that the initial iterate has no impact on the quality of the solution, only the number of iterations is modified as it was observed for the case of analytical integration.

Example (2):

In this example, we consider data computed from the analytical solution defined as follows

$T(x, y) = ln(1 + sinh(1 - y) * sin(x))$. The conductivity is defined as in the first example by the following function $K(T) = exp(T)$. The tolerance in the stopping criterion of CGLS and Newton are taken to be $tol = 10^{-12}$ and we take as initial value of Newton's iteration the function $v_0(x) = ln((4/\pi)cos(\pi x/2) + 1000)$. Figure 3 presents the comparison between the exact solution and the polynomial approximation for different values of m. The plot shows that the approximate solution gets closer and closer to the exact solution as m increases. The best results are obtained for $m = 4$.

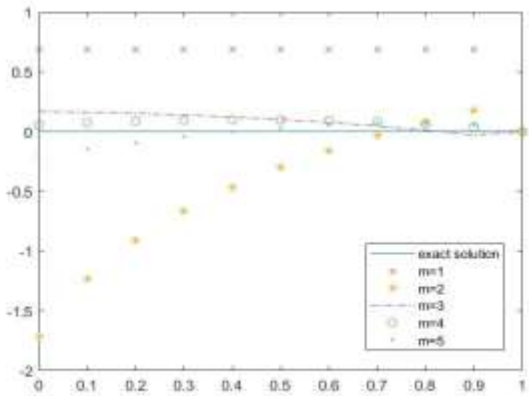

Figure 3: Solutions to Example 2 for various values of m.

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Table 7: Results for Example 2 when $T_0(x) = v_0(x)$.

	Γ	MP	ERROR	ERROR ON ND	ITERN	ITERC	M
No Reg. And No Prec.	-	-	0.257	0.805	67	16	-
Reg. And No Prec.	-	-	0.190	0.802	67	15	1.00e-01
TSkMP	0.3	1	0.092	0.109	67	13	1.00e-02

We also observe that for this example, the solution obtained without regularization without preconditioning remains quite far from the exact solution. We have therefore calculated the solution by introducing the regularization and the preconditioning as they were introduced in [52, 53]. We observe from Table 7 that the quality of the solution has improved significantly for almost the same cost or even a small improvement. The introduction of the regularization combined with the preconditioning allowed a clear improvement of the flux. Indeed, the difference between the calculated normal derivative and the exact normal derivative goes from 80% to 10%. Furthermore, we analyze the computational efficiency of the polynomial expansion method. Figure 4 depicts the computation time as a function of the problem size. The plot demonstrates the rapidity of the method, with computation times scaling favorably even for large problem sizes. This efficiency is a significant advantage of the polynomial expansion method.

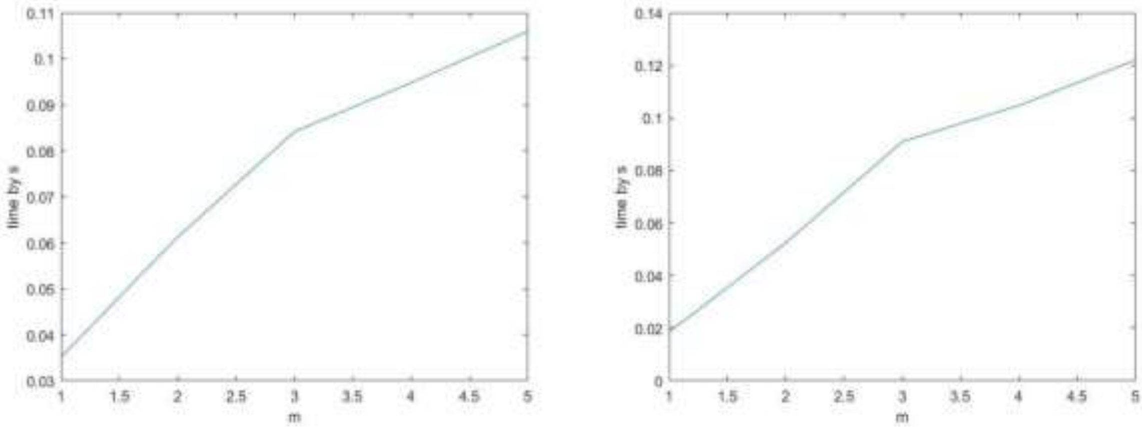

Figure 4: Computation time as a function of the problem size, Example 1 (Left) and Example 2 (Right).

Finally, we conducted an analysis of the solutions obtained with noisy data to evaluate the regularizing character of the technique for solving the nonlinear Cauchy problem. The figures and tables present the results of solving the problem with various levels of noise added to the given data. Figures 5 and 6 illustrate the solutions obtained for different noise levels. It can be observed that the more the noise decreases, the more the accuracy of the solution increases. However, even with significant noise, the methodology demonstrates a remarkable ability to recover the underlying solution. The solutions exhibit smoothness and continuity, indicating the regularizing effect of the polynomial expansion method. The technique effectively suppresses the noise-induced fluctuations (see the noisy data in the left of Fig. 5 and Fig. 6) and produces stable and reliable solutions. These results validate the robustness of the polynomial expansion method with respect to noisy data. The technique exhibits a regularizing effect, preserving the smoothness and stability of the solution despite the presence of noise. It provides a valuable tool for handling real-world scenarios where data is inevitably subject to noise and uncertainties.

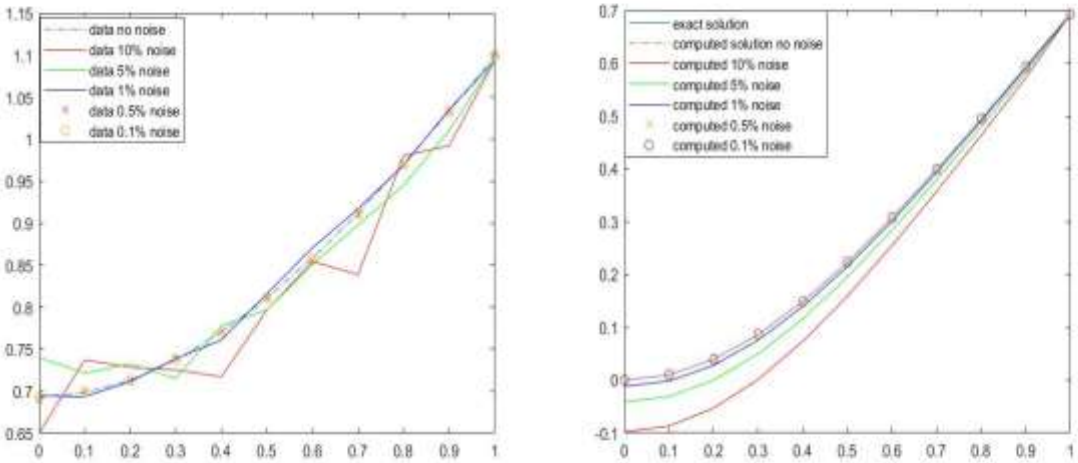

Figure 5: Example 1, Noisy data (left), Computed solution for various noises (right).

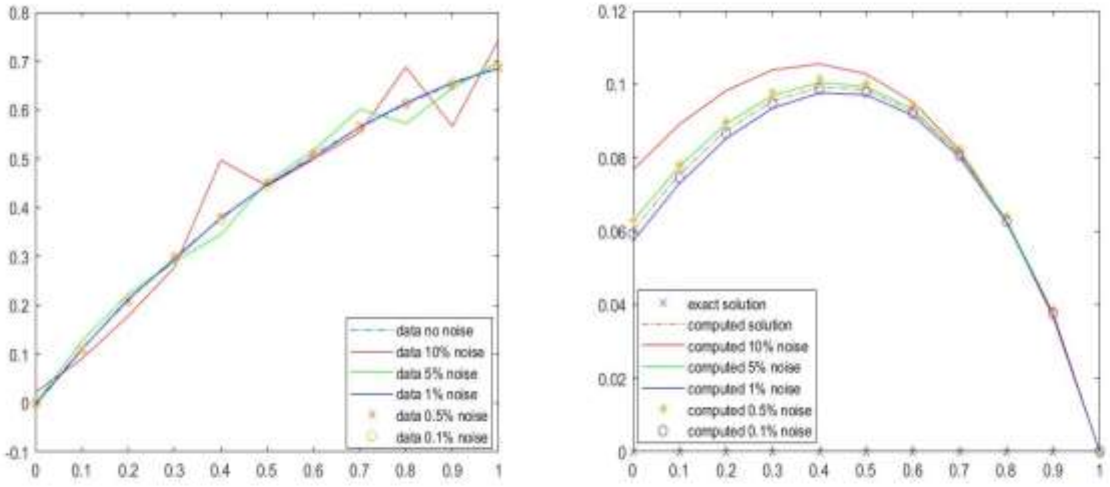

Figure 6: Example 2, Noisy data (left), Computed solution for various noises (right).

In summary, the numerical results highlight the efficiency and accuracy of the polynomial expansion method with the collocation technique for solving the linear Cauchy problem. The rapid convergence of Newton's method and the capability of handling nonlinear equations further enhance the effectiveness of the methodology. The interpretation of the figures validates the performance of the approach and confirms its suitability for a wide range of practical applications. In addition to the figures, we also present tables to further analyze the performance of the methodology. These tables illustrate the influence of the initial iteration on the quality of the solution and its impact on the number of iterations required for convergence.

Tables 2, 4, 5, 6, and 7 show the results obtained for different initial iterations. It can be observed that the quality of the solution remains consistent across various initial guesses. The differences in the solution accuracy are negligible, indicating that the initial guess has no significant influence on the final solution quality. However, there is a notable effect on the number of iterations needed to achieve convergence. A closer examination reveals that an appropriate initial guess can lead to faster convergence, reducing the computational time significantly.

Conclusions

In this study, we presented a methodology based on the polynomial expansion method with the collocation technique for approximating the solution of the linear Cauchy problem. The approach involved expanding the solution $\omega(x, y)$ using a polynomial of degree m, and determining the coefficients by solving a system of equations obtained by evaluating the equations on collocation nodes. The nonlinear scalar equations arising from the expansion were efficiently resolved using Newton's method. The numerical results demonstrated the effectiveness and efficiency of the proposed methodology. The polynomial expansion provided accurate solutions for various test cases, even for complex geometries, without the need for mesh generation. The rapid convergence of Newton's method further enhanced the efficiency of the approach, resulting in accurate approximations of the solution. The interpretation of the figures and tables showcased the robustness of the methodology. The results showed that the initial iteration had no significant influence on the quality of the solution, but played a crucial role in the number of iterations required for convergence. Selecting an appropriate initial guess led to faster convergence, reducing the computational effort required to obtain accurate solutions. The polynomial expansion method with the collocation technique offers several advantages, including its mesh-free nature, suitability for handling nonlinear problems, and parallelizability. These features make it a promising approach for solving inverse boundary problems, particularly in heat transfer applications. Future work could focus on extending the methodology to three-dimensional problems and investigating its applicability to other types of inverse boundary problems. Overall, the presented methodology, combined with the numerical results obtained, demonstrates its efficacy in accurately approximating the solution of the nonlinear Cauchy problem. It provides a valuable tool for researchers and practitioners in the field of inverse boundary problems.

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