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Almost Approximaitly Nearly Semiprime Submodules II

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Abstract

Our goal in this research is to study the concept of almost approximaitly nearly semiprime submodules in class of multiplication modules. Moreover, we study the relationship between almost approximaitly nearly semiprime submodules and their residuals. We characterized almost approximaitly nearly semiprime ideal B by almost approximaitly nearly semiprime submodule of the form Bʜ in class of cyclic R-modules.

Keywords: Multiplication modules, Projective modules, Faithful modules, Content modules, non-singular modules, almost approximaitly nearly semiprime submodules.

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المقاساتالجزئية شبه الاولية المتقارب تقريباً كلياً II

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الخلاصة

هدفنا في هذا البحث هو دراسة مفهوم المقاسات الجزئية شبه المتقاربة تقريبًا كليا في صف المقاسات الجدائية. بالاضافة الى ذالك, ندرس العلاقة بين المقاسات الجزئية شبه المتقاربة تقريبًا كليا والريسديول له. وقمنا بمكافئة المثالي شبه المتقاربة تقريبًا كليا B بالمقاسات الجزئية شبه المتقاربة تقريبًا كليا من النمط BҢ في صف المقاسات الدورية.

كلمات مفتاحية : المقاسات الجدائية، المقاسات الاسقاطية، المقاسات المخلصة، المقاسات المضمونه، المقاسات الغير-منفرده، المقاسات الجزئية شبه الاولية المتقارب تقريباً كلياً.

Introduction

Let Ʀ be commutative ring with identity and Ң be left unitary Ʀ-module. It is worth noting here that this research is a continuation of a study of the concept that was presented in [1], where A proper submodule F of an Ʀ-module Ң is called almost approximaitly nearly semiprime (simply Alappns-prime) submodule, if for any $r^n h \in F$, for $r \in$ Ʀ, $h \in$ Ң, and $n \in Z^+$, implies that $rh \in F + (soc($Ң$) + J($Ң$))$. An ideal J of a ring Ʀ is Alappns-prime ideal of Ʀ if J is an Alappns-prime Ʀ-submodule of an Ʀ-module Ʀ. In particular a proper submodule F of an Ʀ-module Ң is Alappns-prime if for any $r^2 h \in F$, for $r \in$ Ʀ, $h \in$ Ң, implies that $rh \in F + (soc($Ң$) + J($Ң$))$, to which was a generalization of [2, 3, 4, 5, 6]. This paper consists of three sections. Section one covers some basic concepts, recalls some remarks and propositions needed in the sequel. Section two, study the concept of almost approximaitly nearly semiprime submodule in class of multiplication modules and give several characterizations. In addition to studying the relationship between almost approximaitly nearly semiprime submodule and theirs residual, where the residual of submodule F by Ң denoted by $[F:_{Ʀ}$ Ң$] = \{r \in$ Ʀ$: r$Ң $\subseteq F\}$ which is an ideal of Ʀ [7]. Section three devoted to introduce many characterizations of Alappns-prime submodules in class of cyclic R-modules.

Preliminaries

This section includes some well-known definitions, remarks and propositions that will be needed study of the next sections.

Definition 1.1 [7] An Ɍ-module Ḩ is multiplication, if every submodule F of Ḩ is of the form $F = I$Ḩ for some ideal I of Ɍ. Equivalently Ḩ is a multiplication Ɍ-module if $F = [F :_{Ɍ}$ Ḩ]Ḩ.

Definition 1.2 [8] For any submodule F and K of a multiplication R-module Ḩ with $F = I$Ḩ and $K = J$Ḩ for some ideals I and J of R. The product $FK = I$Ḩ.JḨ $= IJ$Ḩ, that is $FK = IK$. In particular FḨ $= I$ḨḨ $= I$Ḩ $= F$. Also for any $x \in$ Ḩ we have $F = Ix$ and $x = Rx$ as a submodule of Ḩ.

Proposition 1.3 [1, Prop. 3.3] Let Ḩ be an Ɍ-module, and $F \subsetneq$ Ḩ. Then F is an Alappns-prime submodule of Ḩ if and only if whenever $I^n L \subseteq F$, for I is an ideal of Ɍ, L is a submodule of Ḩ and $n \in Z^+$, implies that $IL \subseteq F + (soc($Ḩ$) + J($Ḩ$))$.

The following corollary is a direct consequence from proposition 1.3.

Corollary 1.4 Let Ḩ be an Ɍ-module, and $F \subsetneq$ Ḩ. Then F is an Alappns-prime submodule of Ḩ if and only if whenever I^nḨ $\subseteq F$, for I is an ideal of Ɍ, $n \in Z^+$, implies that IḨ $\subseteq F + (soc($Ḩ$) + J($Ḩ$))$.

Definition 1.5 [10] An Ɍ-module Ḩ is projective if every Ɍ-epimorphism f from Ɍ-module Ḩ$'$ into Ɍ-module Ḩ and for any Ɍ-homomorphism g from Ɍ-module Ḩ into Ɍ- module Ḩ there exists an Ɍ-homomorphism h from Ɍ-module Ḩ into Ɍ- module Ḩ$'$ such that $f \circ h = g$.

Proposition 1.6

1. If Ḩ is a projective Ɍ-module, then $soc($Ɍ$)$Ḩ $= soc($Ḩ$)$ [11, Prop. 3.24].
2. If Ḩ is a projective Ɍ-module, then $J($Ɍ$)$Ḩ $= J($Ḩ$)$ [7, Prop. 17.10].

Proposition 1.7 [1, Cor. 3.4] Let Ң be an Ɍ-module, and $F \subsetneq$ Ң. Then F is an Alappns-prime submodule of Ң if and only if whenever $I^2L \subseteq F$, for I is an ideal of Ɍ, and L is a submodule of Ң, implies that $IL \subseteq F + (soc(\text{Ң}) + J(\text{Ң}))$.

Definition 1.8 [10] An Ɍ-module Ң is faithful if $ann_{\text{Ɍ}}(\text{Ң}) = (0)$, where $ann_{\text{Ɍ}}(\text{Ң}) = \{r \in \text{Ɍ}: r\text{Ң} = (0)\}$.

Proposition 1.9

1. If Ң be a faithful multiplication Ɍ-module, then $soc(\text{Ɍ})\text{Ң} = soc(\text{Ң})$ [9, Coro. 2.14(i)].
2. If Ң be a faithful multiplication Ɍ-module, then $J(\text{Ɍ})\text{Ң} = J(\text{Ң})$ [11, Rem. p14].

Proposition 1.10 [1, Cor. 3.7] Let Ң be an Ɍ-module, and $F \subsetneq$ Ң. Then F is an Alappns-prime submodule of Ң if and only if whenever $r^nL \subseteq F$, for $r \in$ Ɍ, L is a submodule of Ң and $n \in Z^+$, implies that $rL \subseteq F + (soc(\text{Ң}) + J(\text{Ң}))$.

Definition 1.11 [12] An Ɍ-module Ң is a non-singular if $Z(\text{Ң}) = (0)$, where $Z(\text{Ң}) = \{x \in \text{Ң}: xI = (0),\ \text{for some essential ideal I of Ɍ}\}$.

Proposition 1.12 [12, Cor. 1.26] Let Ң be non-singular Ɍ-modules, so $soc(\text{Ɍ})\text{Ң} = soc(\text{Ң})$.

Definition 1.13 [13] An Ɍ-module Ң is content module if $\left(\bigcap_{j\in I} B_j\right)\text{Ң} = \bigcap_{j\in I} B_j\,\text{Ң}$ for each family of ideals B_j of Ɍ.

Proposition 1.14 [11, Prop. 1.10] If Ң is content module, then $J(\text{Ɍ})\text{Ң} = J(\text{Ң})$.

Definition 1.15 [10] A ring Ɍ is called a good ring if $J(\text{Ң}) = J(\text{Ɍ})\text{Ң}$ for any Ɍ-module Ң.

Definition 1.16 [14] A ring Ɍ is an Artinian if satisfies descending chain condition (DCC) on ideals of Ɍ.

Proposition 1.17 [10, Coro. 9.7.3(b)] Every Artinian ring is a good ring.

Definition 1.18 [14] A ring Ɍ is local if Ɍ has a unique maximal ideal.

Proposition 1.19 [15, Prop. 1.12] If Ң be an Ɍ-module over local ring Ɍ, then $J(\text{Ң}) = J(\text{Ɍ})\text{Ң}$.

Proposition 1.20 [16, Coro. of Theo. (9)] Let Ң be a finitely generated multiplication Ɍ-module and I, J are ideals of Ɍ. Then $I\text{Ң} \subseteq J\text{Ң}$ if and only if $I \subseteq J + ann_{\text{Ɍ}}(\text{Ң})$.

Proposition 1.21 [17, Prop. 3.1] If Ң is faithful multiplication finitely generated Ɍ-module, then Ң is cancellation.

Definition 1.22 [17] An Ɍ-module Ң is called cancellation if $I\text{Ң} = J\text{Ң}$ for any ideals I, J in Ɍ then $I = J$.

Definition 1.23 [17] An Ɍ- module Ң is called weak cancellation if $I\text{Ң} = J\text{Ң}$, implies that $I + ann_{\text{Ɍ}}(\text{Ң}) = J + ann_{\text{Ɍ}}(\text{Ң})$ for I, J are ideals in Ɍ.

Proposition 1.24 [17, Prop. 3.9] If Ң is a multiplication Ɍ-module, then Ң is finitely generated if and only if Ң is weak cancellation.

Almost Approximately Nearly Semiprime Submodules in multiplication Modules

In this section we introduce the notation of almost approximately nearly semiprime submodules in class of multiplication modules. In addition to studying the relationship between the almost approximately nearly semiprime submodules with the residual of this concept.

Proposition 2.1 Let Ң be multiplication Ɍ-module, and F is proper submodule of Ң. Then F is an Alappns-prime submodule of Ң if and only if whenever $L^n K \subseteq F$ for L, K are submodules of Ң, and $n \in Z^+$, implies that $LK \subseteq F + (soc(\text{Ң}) + J(\text{Ң}))$.

Proof (⟾) Let $L^n K \subseteq F$ for L, K are submodules of Ң, and $n \in Z^+$. Since Ң is a multiplication, then $L^n = (I\text{Ң})^n$, $K = J\text{Ң}$ for some ideals I, J of Ɍ. That is $(I\text{Ң})^n(J\text{Ң}) = I^n(J\text{Ң}) \subseteq F$. But F is an Alappns-prime submodule of Ң, then by proposition 1.3 $I(J\text{Ң}) \subseteq F + (soc(\text{Ң}) + J(\text{Ң}))$. Thus $LK \subseteq F + (soc(\text{Ң}) + J(\text{Ң}))$.

(⟽) Suppose $I^n U \subseteq F$ for I is an ideal of Ɍ and U is a submodule of Ң, and $n \in Z^+$. Since Ң is a multiplication, then $U = J\text{Ң}$ for some ideal J of Ɍ, that is $I^n(J\text{Ң}) \subseteq F$,take $C = I\text{Ң}$, so

$C^nU \subseteq F$. Then by hypothesis, we have $CU \subseteq F + (soc(Ḥ) + J(Ḥ))$. Thus $IU \subseteq F + (soc(Ḥ) + J(Ḥ))$. Hence by proposition 1.3 F is an Alappns-prime submodule of Ḥ.

These corollaries are directly from Proposition 2.1

Corollary 2.2 Let Ḥ be multiplication R-module, and F is proper submodule of Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if whenever ${h_1}^n h_2 \subseteq F$ for $h_1, h_2 \in$ Ḥ, and $n \in Z^+$ implies that $h_1 h_2 \subseteq F + (soc(Ḥ) + J(Ḥ))$.

Corollary 2.3 Let Ḥ be multiplication R-module, and F is proper submodule of Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if whenever $L^n h \subseteq F$ for L is a submodules of Ḥ, $h \in$ Ḥ, and $n \in Z^+$, implies that $Lh \subseteq F + (soc(Ḥ) + J(Ḥ))$.

Corollary 2.4 Let Ḥ be a multiplication R-module, and F is a proper submodule of Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if whenever $h^n U \subseteq F$ for $h \in$ Ḥ, U is a submodules of Ḥ, and $n \in Z^+$, implies that $hU \subseteq F + (soc(Ḥ) + J(Ḥ))$.

Proposition 2.5 Let F be a proper submodule of a multiplication R-module Ḥ. Then the following statements are equivalent:

1. F is an Alappns-prime submodule of Ḥ.

2. $x^n \in F$ implies that $x \in F + (soc(Ḥ) + J(Ḥ))$ for every $x \in$ Ḥ.

3. $\sqrt{F} \subseteq F + (soc(Ḥ) + J(Ḥ))$.

4. $E_1 E_2 \ldots \ldots E_j \subseteq F$, implies that $E_1 \cap E_2 \cap \ldots \ldots \cap E_j \subseteq F + (soc(Ḥ) + J(Ḥ))$ for every submodules $E_1, E_2, \ldots \ldots, E_j$ of Ḥ and $j \in Z^+$.

Proof (1) ⟹(2) Let $x^n \in F$, for $x \in$ Ḥ and $n \in Z^+$, then $\langle x^n \rangle \subseteq F$. But Ḥ is a multiplication R-module, then $x = Rx = I$Ḥ for some ideal I of R, so $x^n = I^n$Ḥ $\subseteq F$. Since F is an Alappns-prime submodule of Ḥ, then by corollary 1.4 IḤ $\subseteq F + (soc(Ḥ) + J(Ḥ))$. That is $x \subseteq F + (soc(Ḥ) + J(Ḥ))$, implies that $x \in F + (soc(Ḥ) + J(Ḥ))$.

(2) ⟹(3) Let $x \in \sqrt{F}$, implies that $x^n \in F$ for some $n \in Z^+$, so by hypothesis $x \in F + soc(Ḥ) + J(Ḥ)$. Thus $\sqrt{F} \subseteq F + (soc(Ḥ) + J(Ḥ))$.

(3) ⟹(4) Suppose that $E_1E_2 \dots\dots E_j \subseteq F$ for $E_1, E_2, \dots\dots, E_j$ are submodules of Ⱨ and $j \in Z^+$. Let $x \in E_1 \cap E_2 \cap \dots\dots \cap E_j$ then $x \in E_i$ for each $i = 1,2, \dots\dots, j$, so $x^j \in E_1E_2 \dots\dots E_j \subseteq F$, it follows that $x^j \in F$, so $x \in \sqrt{F} \subseteq F + (soc(Ⱨ) + J(Ⱨ))$, implies that $x \in F + (soc(Ⱨ) + J(Ⱨ))$. Thus $E_1 \cap E_2 \cap \dots\dots \cap E_j \subseteq F + (soc(Ⱨ) + J(Ⱨ))$.

(4) ⟹(1) Let $I^nⱧ \subseteq F$, where I is an ideal of Ɍ, and $n \in Z^+$. That is $(IⱧ)(IⱧ) \dots\dots (IⱧ) \subseteq F$, so by hypothesis $(IⱧ) \cap (IⱧ) \cap \dots\dots \cap (IⱧ) \subseteq F + (soc(Ⱨ) + J(Ⱨ))$. Implies that $IⱧ \subseteq F + (soc(Ⱨ) + J(Ⱨ))$. Thus by corollary 1.4 F is an Alappns-prime submodule of Ⱨ.

Remark 2.6 The residual of Alappns-prime submodule of an Ɍ-module Ⱨ is not Alappns-prime ideal of Ɍ, as in this example:

Let $Ⱨ = Z_{72}$, $Ɍ = Z$, the submodule $F = \langle \bar{4} \rangle$ is an Alappns-prime submodule of Z_{72}. But $[F :_Z Z_{72}]$=$[\langle \bar{4} \rangle :_Z Z_{72}] = 4Z$ is not Alappns-prime ideal of Z because $2^2.1 \in 4Z$ for $1, 2 \in Z$ but $2 \notin 4Z + (soc(Z) + J(Z)) = 4Z + (0) = 4Z$.

So the following results show that under certain conditions this becomes true.

Proposition 2.7 A proper submodule F of projective multiplication Ɍ-module Ⱨ is an Alappns-prime submodule of Ⱨ if and only if $[F :_Ɍ Ⱨ]$ is an Alappns-prime ideal for Ɍ.

Proof (⟾) Let $I^nJ \subseteq [F :_Ɍ Ⱨ]$, where I, J are ideals in Ɍ, and $n \in Z^+$, implies that $I^nJⱧ \subseteq F$. Since Ⱨ is multiplication, then $I^nJⱧ = L^nK$ by taking $L = IⱧ$, $K = JⱧ$ are submodules of Ⱨ, hence $L^nK \subseteq F$. But F is an Alappns-prime submodule of multiplication Ɍ-module Ⱨ, then by proposition 2.1 $LK \subseteq F + soc(Ⱨ) + J(Ⱨ)$. Again since Ⱨ is multiplication, then $F = [F :_Ɍ Ⱨ]Ⱨ$, and since Ⱨ is projective then by proposition 1.6 $soc(Ⱨ) = soc(Ɍ)Ⱨ$ and $J(Ⱨ) = J(Ɍ)Ⱨ$ Thus $IJⱧ \subseteq [F :_Ɍ Ⱨ]Ⱨ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$, it follows that $IJ \subseteq [F :_Ɍ Ⱨ] + soc(Ɍ) + J(Ɍ)$. Therefor by proposition 1.3 $[F :_Ɍ Ⱨ]$ is an Alappns-prime ideal of Ɍ.

(⟽) Let $I^nK \subseteq F$, for I is an ideal of Ɍ and K is submodule of Ⱨ, and $n \in Z^+$. Since Ⱨ is multiplication, so $K = JⱧ$ for some ideal J in Ɍ, that is $I^nJⱧ \subseteq F$, follow it $I^nJ \subseteq [F :_Ɍ Ⱨ]$, but $[F :_Ɍ Ⱨ]$ is an Alappns-prime ideal of Ɍ, then by proposition 1.3 $IJ \subseteq [F :_Ɍ Ⱨ] + soc(Ɍ) + J(Ɍ)$.

Hence $IJⱧ \subseteq [F:_{Ɍ} Ⱨ]Ⱨ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$. Thus by proposition 1.6 $IJⱧ \subseteq F + soc(Ⱨ) + J(Ⱨ)$. That is $IK \subseteq F + soc(Ⱨ) + J(Ⱨ)$. Thus by proposition 1.3 F is an Alappns-prime submodule for Ⱨ.

Proposition 2.8 A proper submodule F of faithful multiplication Ɍ-module Ⱨ is an Alappns-prime submodule of Ⱨ if and only if $[F:_{Ɍ} Ⱨ]$ is an Alappns-prime ideal of Ɍ.

Proof ($\Longrightarrow$) Suppose that F is an Alappns-prime submodule of Ⱨ, and Let $r^2 I \subseteq [F:_{Ɍ} Ⱨ]$ for $r \in Ɍ$, and I is an ideal of Ɍ, implies that $r^2(IⱧ) \subseteq F$. But F is Alappns-prime submodule of Ⱨ, so we have by proposition 1.7 $rIⱧ \subseteq F + soc(Ⱨ) + J(Ⱨ)$. Since Ⱨ is multiplication, then $F = [F:_{Ɍ} Ⱨ]Ⱨ$, and since Ⱨ is faithful multiplication, then by proposition 1.9 $soc(Ⱨ) = soc(Ɍ)Ⱨ$ and $J(Ⱨ) = J(Ɍ)Ⱨ$. Thus $rIⱧ \subseteq [F:_{Ɍ} Ⱨ]Ⱨ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$, it follows that $rI \subseteq [F:_{Ɍ} Ⱨ] + soc(Ɍ) + J(Ɍ)$. Hence by proposition 1.7 $[F:_{Ɍ} Ⱨ]$ is an Alappns-prime ideal of Ɍ.

($\Longleftarrow$) Let $r^2 K \subseteq F$ for $r \in Ɍ$ and K is a submodule of Ⱨ. Since Ⱨ is multiplication, so $K = JⱧ$ for some ideal J in Ɍ, that is $r^2 JⱧ \subseteq F$, follow it $r^2 J \subseteq [F:_{Ɍ} Ⱨ]$, but $[F:_{Ɍ} Ⱨ]$ is an Alappns-prime ideal for Ɍ, then by proposition 1.7 $rJ \subseteq [F:_{Ɍ} Ⱨ] + soc(Ɍ) + J(Ɍ)$. Hence $rJⱧ \subseteq [F:_{Ɍ} Ⱨ]Ⱨ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$. Hence by proposition 1.9 $rJⱧ \subseteq F + soc(Ⱨ) + J(Ⱨ)$. That is $rK \subseteq F + soc(Ⱨ) + J(Ⱨ)$. Thus by proposition 1.7 F is an Alappns-prime submodule of Ⱨ.

Proposition 2.9 A proper submodule F of a content multiplication non-singular Ɍ-module Ⱨ is an Alappns-prime submodule of Ⱨ if and only if $[F:_{Ɍ} Ⱨ]$ is an Alappns-prime ideal of Ɍ.

Proof ($\Longrightarrow$) Let $r^n s \in [F:_{Ɍ} Ⱨ]$ for $r, s \in Ɍ$, and $n \in Z^+$, so $r^n(sⱧ) \subseteq F$. But F is an Alappns-prime submodule of Ⱨ, then by proposition 1.10 $r(sⱧ) \subseteq F + soc(Ⱨ) + J(Ⱨ)$. Since Ⱨ is multiplication, then $F = [F:_{Ɍ} Ⱨ]Ⱨ$, and since Ⱨ is non-singular multiplication, then by proposition 1.12 $soc(Ⱨ) = soc(Ɍ)Ⱨ$ and Ⱨ is content Ɍ-module then by proposition 1.14)$J(Ⱨ) = J(Ɍ)Ⱨ$. Thus. $rsⱧ \subseteq [F:_{Ɍ} Ⱨ]Ⱨ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$, it follows that $rs \in [F:_{Ɍ} Ⱨ] + soc(Ɍ) + J(Ɍ)$. Hence $[F:_{Ɍ} Ⱨ]$ is an Alappns-prime ideal of Ɍ.

($\Longleftarrow$) Let $L^n h \subseteq F$ for L is a submodule of Ⱨ, $h \in Ⱨ$, and $n \in Z^+$. Since Ⱨ is a multiplication, then $L = IⱧ$ and $h = Ɍh = JⱧ$ for some ideals I, J of Ɍ, that is $I^n JⱧ \subseteq F$, implies that $I^n J \subseteq$

$[F:_{Ɍ} Ḩ]$, but $[F:_{Ɍ} Ḩ]$ is an Alappns-prime ideal of Ɍ, then by proposition 1.3 $IJ \subseteq [F:_{Ɍ} Ḩ] + soc(Ɍ) + J(Ɍ)$. Hence $IJḨ \subseteq [F:_{Ɍ} Ḩ]Ḩ + soc(Ɍ)Ḩ + J(Ɍ)Ḩ$. Hence by proposition 1.12 and proposition 1.14 $IJḨ \subseteq F + soc(Ḩ) + J(Ḩ)$. That is $Lh \subseteq F + soc(Ḩ) + J(Ḩ)$. Thus by corollary 2.3 F is an Alappns-prime submodule of Ḩ.

Proposition 2.10 Let Ḩ be a non-singular multiplication module over a good ring Ɍ, and $F \subset$ Ḩ. Then F is an Alappns-prime submodule of Ḩ if and only if $[F:_{Ɍ} Ḩ]$ is an Alappns-prime ideal of Ɍ.

Proof ($\Longmapsto$) Let $I^n J \in [F:_{Ɍ} Ḩ]$ for I, J are ideals in Ɍ and $n \in Z^+$, implies that $I^n(JḨ) \subseteq F$. But F is an Alappns-prime submodule of Ḩ, then by proposition 2.1 $IJḨ \subseteq F + soc(Ḩ) + J(Ḩ)$. Since Ḩ is multiplication, then $F = [F:_{Ɍ} Ḩ]Ḩ$, and since Ḩ is non-singular multiplication, then by proposition 1.12 $soc(Ḩ) = soc(Ɍ)Ḩ$ and Ɍ is a good ring then $J(Ḩ) = J(Ɍ)Ḩ$. Thus $IJḨ \subseteq [F:_{Ɍ} Ḩ]Ḩ + soc(Ɍ)Ḩ + J(Ɍ)Ḩ$, it follows that $IJ \subseteq [F:_{Ɍ} Ḩ] + soc(Ɍ) + J(Ɍ)$. Hence by proposition 2.1 $[F:_{Ɍ} Ḩ]$ is Alappns-prime ideal for Ɍ.

($\Longleftarrow$) Let ${h_1}^n h_2 \subseteq F$ for $h_1, h_2 \in$ Ḩ, and $n \in Z^+$. Since Ḩ is a multiplication, then $h_1 = Ɍh_1 = IḨ$, $h_2 = Ɍh_2 = JḨ$ for some ideals I, J of Ɍ, that is $IJḨ \subseteq F$, implies that $I^n J \subseteq [F:_{Ɍ} Ḩ]$, but $[F:_{Ɍ} Ḩ]$ is an Alappns-prime ideal of Ɍ, then by proposition 1.3 $IJ \subseteq [F:_{Ɍ} Ḩ] + soc(Ɍ) + J(Ɍ)$. Hence $IJḨ \subseteq [F:_{Ɍ} Ḩ]Ḩ + soc(Ɍ)Ḩ + J(Ɍ)Ḩ$. Hence by proposition 1.12 and Ɍ is a good ring $IJḨ \subseteq F + soc(Ḩ) + J(Ḩ)$. That is $h_1 h_2 \subseteq F + soc(Ḩ) + J(Ḩ)$. Thus by corollary 2.2 F is an Alappns-prime submodule for Ḩ.

Since by proposition 1.17 Artinian ring is good ring. We get this following result from proposition 2.10.

Corollary 2.11 Let Ḩ be a non-singular multiplication module over Artinian ring Ɍ, and F be a proper submodule of Ḩ. Then F is an Alappns-prime submodule of Ḩ if and only if $[F:_{Ɍ} Ḩ]$ is an Alappns-prime ideal of Ɍ.

Proposition 2.12 Let Ḥ be a non-singular multiplication Ɍ-module over a local ring Ɍ, and F be a proper submodule of Ḥ. Then F is an Alappns-prime submodule of Ḥ if and only if $[F :_{Ɍ} Ḥ]$ is an Alappns-prime ideal of Ɍ.

Proof In the same way as the proof of proposition 2.10 by using proposition 1.19.

Different Characterizations of Almost Approximately Nearly Semiprime Submodules in Class of Cyclic Module

In this section we introduce many characterizations of Alappns-prime ideals with special kind of Alappns-prime submodules in some types of modules.

Proposition 3.1 Let Ḥ be cyclic projective Ɍ-module, and B is an ideal of Ɍ such that $ann_{Ɍ}(Ḥ) \subseteq B$. Then B is an Alappns-prime ideal of Ɍ if and only if $BḤ$ is an Alappns-prime submodule of Ḥ.

Proof (⟾) Let $L^n K \subseteq BḤ$, for L, K are submodules of Ḥ, and $n \in Z^+$. Since Ḥ is cyclic then Ḥ is multiplication [18], hence $L^n = (IḤ)^n$, $K = JḤ$ for some ideals I, J of Ɍ. That is $(IḤ)^n(JḤ) = I^n(JḤ) \subseteq BḤ$. Again Ḥ is cyclic then Ḥ is a finitely generated [19], hence by proposition 1.20 $I^n J \subseteq B + ann_{Ɍ}(Ḥ)$, but $ann_{Ɍ}(Ḥ) \subseteq B$, implies that $B + ann_{Ɍ}(Ḥ) = B$, thus $I^n J \subseteq B$. Now, by assumption B is an Alappns-prime ideal of Ɍ then by proposition 1.3 $IJ \subseteq B + soc(Ɍ) + J(Ɍ)$, we have $IJḤ \subseteq BḤ + soc(Ɍ)Ḥ + J(Ɍ)Ḥ$. But Ḥ is a projective then by proposition 1.6 $soc(Ḥ) + J(Ḥ) = soc(Ɍ)Ḥ + J(Ɍ)Ḥ$, this leads to $KL \subseteq BḤ + soc(Ḥ) + J(Ḥ)$. Therefore by proposition 2.1 $BḤ$ is an Alappns-prime submodule of Ḥ.

(⟽) Let $I^n J \subseteq B$, for I and J are ideals in Ɍ, and $n \in Z^+$, implies that $I^n(JḤ) \subseteq BḤ$. But $BḤ$ is an Alappns-prime submodule of Ḥ, then by proposition 1.3 $IJḤ \subseteq BḤ + soc(Ḥ) + J(Ḥ)$. But Ḥ is projective then $soc(Ḥ) + J(Ḥ) = soc(Ɍ)Ḥ + J(Ɍ)Ḥ$. So $IJḤ \subseteq BḤ + soc(Ɍ)Ḥ + J(Ɍ)Ḥ$, it follows that $IJ \subseteq B + soc(Ɍ) + J(Ɍ)$. Hence by proposition 1.3 B is an Alappns-prime ideal of Ɍ.

Proposition 3.2 Let Ⱨ be a non-singular cyclic Ɍ-module over a good ring Ɍ, and B is an ideal of Ɍ such that $ann_{Ɍ}(Ⱨ) \subseteq B$. Then B is an Alappns-prime ideal of Ɍ if and only if $BⱧ$ is an Alappns-prime submodule of Ⱨ.

Proof (⟹) Let ${h_1}^n h_2 \subseteq BⱧ$, for $h_1, h_2 \in Ⱨ$, and $n \in Z^+$. Since Ⱨ is a cyclic then Ⱨ is multiplication [18], hence $(h_1)^n = (Ɍh_1)^n = (IⱧ)^n = I^nⱧ$, $h_2 = Ɍh_2 = JⱧ$ for some ideals I, J of Ɍ, that is $I^nJⱧ \subseteq BⱧ$. Again Ⱨ is cyclic then Ⱨ is a finitely generated [19], hence by proposition 1.20 $I^nJ \subseteq B + ann_{Ɍ}(Ⱨ)$, since $ann_{Ɍ}(Ⱨ) \subseteq B$, implies that $B + ann_{Ɍ}(Ⱨ) = B$ implies that $I^nJ \subseteq B$. But B is an Alappns-prime ideal of Ɍ then by proposition 1.3 $IJ \subseteq B + soc(Ɍ) + J(Ɍ)$. Thus $IJⱧ \subseteq BⱧ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$. Since Ⱨ is non-singular, then by proposition 1.12 $soc(Ɍ)Ⱨ = soc(Ⱨ)$ and Ɍ is good ring then $J(Ɍ)Ⱨ = J(Ⱨ)$. We have $IJⱧ \subseteq BⱧ + soc(Ⱨ) + J(Ⱨ)$. That is $h_1h_2 \subseteq BⱧ + soc(Ⱨ) + J(Ⱨ)$. Therefore by corollary 2.2 $BⱧ$ is an Alappns-prime submodule of Ⱨ.

(⟸) Let $r^nI \subseteq B$, for $r \in Ɍ$, and I is an ideal of Ɍ, and $n \in Z^+$, implies that $r^n(IⱧ) \subseteq BⱧ$. Since $BⱧ$ is an Alappns-prime submodule of Ⱨ, then by proposition 1.10 $rIⱧ \subseteq BⱧ + soc(Ⱨ) + J(Ⱨ)$. But Ⱨ is non-singular and Ɍ is good ring then $soc(Ⱨ) + J(Ⱨ) = soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$. Hence $rJⱧ \subseteq BⱧ + soc(Ɍ)Ⱨ + J(Ɍ)Ⱨ$. That is $rI \subseteq BⱧ + soc(Ɍ) + J(Ɍ)$. Therefore by proposition 1.10 B is an Alappns-prime ideal of Ɍ.

Corollary 3.3 Let Ⱨ be a non-singular cyclic Ɍ-module over an Artinian ring Ɍ, and B is an ideal of Ɍ such that $ann_{Ɍ}(Ⱨ) \subseteq B$. Then B is an Alappns-prime ideal of Ɍ if and only if $BⱧ$ is an Alappns-prime submodule of Ⱨ.

Proposition 3.4 Let Ⱨ be a non-singular cyclic Ɍ-module over local ring Ɍ, and B is an ideal of Ɍ such that $ann_{Ɍ}(Ⱨ) \subseteq B$. Then B is an Alappns-prime ideal of Ɍ if and only if $BⱧ$ is an Alappns-prime submodule of Ⱨ.

Proof Follows in the same way as the proof of proposition 3.2 by using proposition 1.19.

Proposition 3.5 Let Ⱨ be a cyclic faithful Ɍ-module, and B be an Alappns-prime ideal of Ɍ. Then $BⱧ$ is an Alappns-prime submodule of Ⱨ.

Proof (⟹) Let $h^n U \subseteq B$Ⱨ, for $h \in$ Ⱨ, U is a submodule of Ⱨ, and $n \in Z^+$. Since Ⱨ is a cyclic then Ⱨ is multiplication [18], hence $h^n = (\mathrm{R}h)^n = (I$Ⱨ$)^n = I^n$Ⱨ, $U = J$Ⱨ for some ideals I, J of Ɍ, that is $I^n J$Ⱨ $\subseteq B$Ⱨ. Again Ⱨ is cyclic then Ⱨ is a finitely generated [19], hence by proposition 1.20 $I^n J \subseteq B + ann_{\mathrm{R}}($Ⱨ$)$, since $ann_{\mathrm{R}}($Ⱨ$) \subseteq B$, implies that $B + ann_{\mathrm{R}}($Ⱨ$) = B$implies that $I^n J \subseteq B$. But B is an Alappns-prime ideal of Ɍ then by proposition 1.3 $IJ \subseteq B + soc(\mathrm{R}) + J(\mathrm{R})$. Thus IJⱧ $\subseteq B$Ⱨ $+ soc(\mathrm{R})$Ⱨ $+ J(\mathrm{R})$Ⱨ. But Ⱨ is faithful multiplication, then by proposition 1.9 $soc($Ⱨ$) = soc(\mathrm{R})$Ⱨ and $J($Ⱨ$) = J(\mathrm{R})$Ⱨ. Hence IJⱧ $\subseteq B$Ⱨ $+ soc($Ⱨ$) + J($Ⱨ$)$. That is $hU \subseteq B$Ⱨ $+ (soc($Ⱨ$) + J($Ⱨ$))$. Therefore by corollary 2.4 BⱧ is an Alappns-prime submodule of Ⱨ.

(⟸) Let $r^n s \in B$, for $r, s \in$ Ɍ, and $n \in Z^+$, implies that $r^n(s$Ⱨ$) \subseteq B$Ⱨ. But BⱧ is an Alappns-prime submodule of Ⱨ, then by proposition 1.10 rsⱧ $\subseteq B$Ⱨ $+ soc($Ⱨ$) + J($Ⱨ$)$. Hence by proposition 1.9 rsⱧ $\subseteq B$Ⱨ $+ soc(\mathrm{R})$Ⱨ $+ J(\mathrm{R})$Ⱨ, it follows that $rs \in B + soc(\mathrm{R}) + J(\mathrm{R})$. Therefore B is an Alappns-prime ideal of Ɍ.

Proposition 3.6 Let Ⱨ be a cyclic faithful Ɍ-module, and $F \subset$ Ⱨ then the following are equivalent:

1. F is an Alappns-prime submodule of Ⱨ.
2. $[F :_{\mathrm{R}}$ Ⱨ$]$ is an Alappns-prime ideal of Ɍ.
3. $F = B$Ⱨ for some Alappns-prime ideal B for Ɍ.

Proof (1) ⟺ (2) Since every cyclic module is multiplication [18], so it follows by proposition 2.8.

(2) ⟹ (3) Assume that $[F :_{\mathrm{R}}$ Ⱨ$]$ is an Alappns-prime ideal of Ɍ. Since Ⱨ is a cyclic then Ⱨ is multiplication [18], hence $F = [F :_{\mathrm{R}}$ Ⱨ$]$Ⱨ, put $B = [F :_{\mathrm{R}}$ Ⱨ$]$ is an Alappns-prime ideal of Ɍ and $F = B$Ⱨ.

(3) ⟹ (1) Assume that $F = B$Ⱨ …..(1) for some an Alappns-prime ideal B of Ɍ. But Ⱨ is a multiplication, then $F = [F :_{\mathrm{R}}$ Ⱨ$]$Ⱨ…..(2), from (1) and (2) we have $[F :_{\mathrm{R}}$ Ⱨ$]$Ⱨ $= B$Ⱨ . Since Ⱨ is a cyclic then Ⱨ is finitely generated [19] but Ⱨ is faithful, then by proposition 1.21 Ⱨ is cancellation, implies that $B = [F :_{\mathrm{R}}$ Ⱨ$]$, hence $[F :_{\mathrm{R}}$ Ⱨ$]$ is an Alappns-prime ideal of Ɍ.

Proposition 3.7 Let Ḩ be cyclic projective Ɍ-module, and $F \subset$ Ḩ with $ann_{Ɍ}(Ḩ) \subseteq [F{:}_{Ɍ}\,Ḩ]$ then the following are equivalent:

1. F is an Alappns-prime submodule of Ḩ.
2. $[F{:}_{Ɍ}\,Ḩ]$ is an Alappns-prime ideal of Ɍ.
3. $F = BḨ$ for some Alappns-prime ideal B of Ɍ with $ann_{Ɍ}(Ḩ) \subseteq B$.

Proof (1) ⇔ (2) It follows by proposition 2.7.

(2) ⟹ (3) Assume that $[F{:}_{Ɍ}\,Ḩ]$ is an Alappns-prime ideal of Ɍ. Since Ḩ is a multiplication, then $F = [F{:}_{Ɍ}\,Ḩ]Ḩ = BḨ$, where $B = [F{:}_{Ɍ}\,Ḩ]$ is an Alappns-prime ideal of Ɍ with $ann_{Ɍ}(Ḩ) = [0{:}_{Ɍ}\,Ḩ] \subseteq [F{:}_{Ɍ}\,Ḩ] = B$, implies that $ann_{Ɍ}(Ḩ) \subseteq B$.

(3) ⟹ (1) Assume that $F = BḨ$ …..(1) for some an Alappns-prime ideal B of Ɍ with $ann_{Ɍ}(Ḩ) \subseteq B$. Since Ḩ is a cyclic then Ḩ is multiplication [18], hence $F = [F{:}_{Ɍ}\,Ḩ]Ḩ$…..(2), from (1) and (2) we have $[F{:}_{Ɍ}\,Ḩ]Ḩ = BḨ$. Since Ḩ is cyclic then Ḩ finitely generated [19], hence by proposition 1.24 Ḩ is weak cancellation, it follows that $[F{:}_{Ɍ}\,Ḩ] + ann_{Ɍ}(Ḩ) = B + ann_{Ɍ}(Ḩ)$, but $ann_{Ɍ}(Ḩ) \subseteq B$, and $ann_{Ɍ}(Ḩ) \subseteq [F{:}_{Ɍ}\,Ḩ]$ implies that $ann_{Ɍ}(Ḩ) + B = B$ and $[F{:}_{Ɍ}\,Ḩ] + ann_{Ɍ}(Ḩ) = [F{:}_{Ɍ}\,Ḩ]$. Thus $B = [F{:}_{Ɍ}\,Ḩ]$, but B is an Alappns-prime ideal of Ɍ, hence $[F{:}_{Ɍ}\,Ḩ]$ is an Alappns-prime ideal of Ɍ.

Proposition 3.8 Let Ḩ be a non-singular cyclic Ɍ-module over (good, Artinian, and local) ring Ɍ, and $F \subset$ Ḩ with $ann_{Ɍ}(Ḩ) \subseteq [F{:}_{Ɍ}\,Ḩ]$**.** Then the following are equivalent:

1. F is an Alappns-prime submodule of Ḩ.
2. $[F{:}_{Ɍ}\,Ḩ]$ is an Alappns-prime ideal of Ɍ.
3. $F = BḨ$ for some Alappns-prime ideal B of Ɍ with $ann_{Ɍ}(Ḩ) \subseteq B$.

Proof (1) ⇔ (2) It follows by proposition [2.10, 2.12, and 2.13].

(2) ⇔ (3) Follows in the same way as the proof of proposition 3.7.

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Conclusion

The main results of this paper are:

- Let F be a proper submodule of a multiplication Ɽ-module Ң. Then the following statements are equivalent:
  1. F is an Alappns-prime submodule of Ң.
  2. $x^n \in F$ implies that $x \in F + (soc(Ң) + J(Ң))$ for every $x \in Ң$.
  3. $\sqrt{F} \subseteq F + (soc(Ң) + J(Ң))$.
  4. $E_1 E_2 \dots \dots E_j \subseteq F$, implies that $E_1 \cap E_2 \cap \dots \dots \cap E_j \subseteq F + (soc(Ң) + J(Ң))$ for every submodules $E_1, E_2, \dots \dots, E_j$ of Ң and $j \in Z^+$.
- A proper submodule F of projective multiplication Ɽ-module Ң is an Alappns-prime submodule of Ң if and only if $[F :_{Ɽ} Ң]$ is an Alappns-prime ideal for Ɽ.
- Let Ң be a cyclic faithful Ɽ-module, and $F \subset Ң$ then the following are equivalent:
  1. F is an Alappns-prime submodule of Ң.
  2. $[F :_{Ɽ} Ң]$ is an Alappns-prime ideal of Ɽ.
  3. $F = BҢ$ for some Alappns-prime ideal B for Ɽ.

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