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Extend Nearly Pseudo-2-Absorbing Submodules

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Abstract

In this paper, the concept of Extend Nearly Pseudo-2-Absorbing submodule has been presented as a generalization of 2-absorbing, nearly-2-absorbing and pseudo-2-absorbing submodules. The characterization and examples of the proposed generalization are given, as well as several properties of suggested concept are proven.

Keywords: 2-absorbing submodule, essential submodule, maximal submodule and multiplication module.

المقاسات الجزئية المستحوذة من النمط ٢ - الكاذبة تقريباً الموسعة

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الخلاصة

في هذه الدراسة، تم تقديم مفهوم المقاسات الجزئية المستحوذة من النمط ٢-الكاذبة تقريباً الموسعة كتعميم للمقاسات الجزئية المستحوذة من النمط -٢، المقاسات الجزئية المستحوذة من النمط ٢-تقريباً والمقاسات الجزئية المستحوذة من النمط ٢- الكاذبة. تم تقديم توصيف وأمثلة لتعميم، بالاضافة الى برهان العديد من الخصائص المختلفة للمفهوم المقترح.

كلمات مفتاحية: المقاسات الجزئية المستحوذة من النمط ٢، المقاس الجزئي الجوهري، المقاس الجزئي الاعظمي والمقاس الجدائي.

Introduction

In this paper, we will denote a commutative ring with identity by R and we let M to be denoted a unitary R-module. A proper submodule H of R-module M is called 2-absorbing if whenever $rsn \in H$, for all $r, s \in R$, $n \in$ M, then $rn \in H$ or $sn \in H$ or rsM $\subseteq H$ [1]. Recently, the notion of nearly-2-absorbing submodule was introduced in 2018 as a proper submodule H of M such that if whenever $rsn \in H$, for all $r, s \in R$,$n \in$ M, then $rn \in H + J$(M) or $sn \in H + J$(M) or rsM $\subseteq H + J$(M)[2], where J(M) is the Jacobson radical of M. The idea of pseudo-2-absorbing submodules was introduced in 2019 as a broad statement of 2-absorbing submodule as a proper submodule H of M such that if whenever $rsn \in H$, for all $r, s \in R$,$n \in$ M, then $rn \in H + soc$(M) or $sn \in H + soc$(M) or rsM $\subseteq H + soc$(M)[3], where soc(M) is the intersection of all essential submodule of M and a nonzero submodule K of M is essential in M if $K \cap E \neq (0)$ for any nonzero submodule E of M [4]. Recalled that a module M is a multiplication if for any submodule H of M, then H=IM for some ideal I of Rm rquivalently, $H = [\, H :_R$ M]M [5]. An R-module M is called faithful if and only if ann_R(M) $= \{0\}$[6].

Results

Definition 1 A proper submodule H of R-module M is said to be Extend Nearly Pseudo-2-Absorbing (for short EXNP-2-Absorbing) submodule of M if every $rsx \in H$, where r,$s \in R$, $x \in$ M indicates that either $rx \in H + soc$(M) $+ J$(M) or $sx \in H + soc$(M) $+ J$(M) or rsM $\subseteq H + soc$(M) $+ J$(M).

An ideal I of a ring R is called EXNP-2-Absorbing ideal of R, if I is an EXNP-2-Absorbing R-submodule of an R-module R.

Remarks and Examples 2

1. Let M $= Z_{48}$, $R = Z$, then the submodule $H = \langle\bar{2}\rangle$ is EXNP-2-Absorbing submodule of M, since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle \cap Z_{48} = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. That is for all $r, s \in Z$ and $n \in Z_{48}$ such that $rsn \in \langle\bar{2}\rangle$, implies that either $rn \in \langle\bar{2}\rangle + soc(Z_{48}) + J(Z_{48}) =$

$\langle\bar{2}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $sn \in \langle\bar{2}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\bar{2}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $rs \in [\langle\bar{2}\rangle + soc(Z_{48}) + J(Z_{48}):_R Z_{48}] = 2Z$. That is $2.1.\bar{1} \in \langle\bar{2}\rangle$, implies that $2.\bar{1} = \bar{2} \in \langle\bar{2}\rangle$ and $2.1 = 2 \in [\langle\bar{2}\rangle + soc(Z_{48}) + J(Z_{48}):_R Z_{48}]$.

2. Clearly, every 2-absorbing submodule of M is EXNP-2-Absorbing submodule, the converse is not true in general, for example:

Let $M = Z_{48}$, $R = Z$, then the submodule $H = \langle\bar{8}\rangle$ is EXNP-2-Absorbing submodule of M, since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{4}\rangle \cap \langle\bar{8}\rangle \cap Z_{48} = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. That is for all $r, s \in Z$ and $n \in Z_{48}$ such that $rsn \in \langle\bar{8}\rangle$, implies that either $rn \in \langle\bar{8}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\bar{8}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $sn \in \langle\bar{8}\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\bar{8}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $rs \in [\langle\bar{8}\rangle + soc(Z_{48}) + J(Z_{48}):_R Z_{48}] = 2Z$. But H is not 2-absorbing, since$2.2.\bar{2} \in \langle\bar{8}\rangle$, for $2 \in Z$ and $\bar{2} \in Z_{48}$, it means that $2.\bar{2} = \bar{4} \notin \langle\bar{8}\rangle$ and $2.2 = 4 \notin [\langle\bar{8}\rangle:_R Z_{48}] = 8Z$.

3. The intersection of two EXNP-2-Absorbing submodules of M not necessarily an EXNP-2-Absorbing submodule of M, note the example:

In the Z-module Z, the submodules $3Z$ and $4Z$ are EXNP-2-Absorbing submodules of Z (since they are 2-absorbing submodules), but $3Z \cap 4Z = 12Z$ is not EXNP-2-Absorbing submodule of the Z-module Z, since if $2.3.2 \in 12Z$, but $2.2 = 4 \notin 12Z + soc(Z) + J(Z)$ and $3.2 = 6 \notin 12Z + soc(Z) + J(Z)$ and $2.3 = 6 \notin [12Z + soc(Z) + J(Z): Z] = 12Z$, since $soc(Z) = (0)$ and $J(Z) = 0$.

4. If H is an EXNP-2-Absorbing submodule of M, then $[H :_R M]$ need not necessarily to be an EXNP-2-Absorbing submodule of M, note the example:

Let $M = Z_{48}$, $R = Z$ and the submodule $H = \langle\overline{24}\rangle$ is EXNP-2-Absorbing submodule of M, since $soc(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle \cap \langle\bar{8}\rangle \cap Z_{48} = \langle\bar{8}\rangle$ and $J(Z_{48}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. Then $\langle 24\rangle + soc(Z_{48}) + J(Z_{48}) = \langle\overline{24}\rangle + \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$, hence for all $r, s \in Z$ and $n \in Z_{48}$ such that $rsn \in \langle\overline{24}\rangle$, implies that either $rn \in \langle\bar{2}\rangle$ or $sn \in \langle\bar{2}\rangle$ or $rs \in 2Z$. But $[\langle\overline{24}\rangle:_Z Z_{48}] = 24Z$ is not an EXNP-2-Absorbing ideal of Z, since $4.3.2 \in 24Z$, for $4,3,2 \in Z$, implies that $4.2 \notin 24Z$ and $3.2 \notin 24Z$ and $3.4 \notin 24Z$.

5. It is obvious that every nearly-2-absorbing submodule of M is EXNP-2-Absorbing submodule, the converse is not true in general, for example:

Take a look at the Z-module Z_{60} and the submodule $H = \langle\overline{30}\rangle$, we see that the only essential submodules of Z_{60} are Z_{60} itself and the submodule $\langle\bar{2}\rangle$, so that $soc(Z_{60}) = Z_{60} \cap \langle\bar{2}\rangle = \langle\bar{2}\rangle$. The only maximal submodules $\langle\bar{2}\rangle$, $\langle\bar{3}\rangle$ and $\langle\bar{5}\rangle$. So that $(Z_{60}) = \langle\overline{30}\rangle$, hence $\langle\overline{30}\rangle$ is EXNP-2-Absorbing submodule of Z_{60}, however nearly-2-Absorbing submodule of Z_{60}, because $2.3.5 \in H$, for $2,3,5 \in Z$, implies that $2.5 \notin H + J(Z_{60}) = \langle\overline{30}\rangle + \langle\overline{30}\rangle = \langle\overline{30}\rangle$ and $3.5 \notin \langle\overline{30}\rangle$ and $2.3 \notin 30Z$.

6. Clearly, every pseudo-2-absorbing submodule of M is EXNP-2-Absorbing submodule, the converse is not true in general, for example:

Let $M = Z_{48}$, $R = Z$ and the submodule $H = \langle\bar{8}\rangle$ is EXNP-2-Absorbing submodule of M, see(remarks and examples 2**)**, however pseudo-2-absorbing submodule of Z_{48}, since $2.2.\bar{2} \in \langle\bar{8}\rangle$, for $2 \in Z$ and $\bar{2} \in Z_{48}$, imply that $2.\bar{2} = \bar{4} \notin N + soc(Z_{48}) = \langle\bar{8}\rangle + \langle\bar{8}\rangle = \langle\bar{8}\rangle$ and $2.2 = 4 \notin [\langle\bar{8}\rangle + \langle\bar{8}\rangle :_R Z_{48}] = 8Z$.

Proposition 3 A proper submodule H of M is EXNP-2-Absorbing submodule of M if and only if for any $r , s \in R$ such that $rs \notin [H + soc(M) + J(M) :_R M]$ we have $[H :_M rs] \subseteq [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$.

Proof

($\Rightarrow$) Suppose that H is EXNP-2-Absorbing submodule of M and let $e \in [H :_M rs]$, then $rse \in H$. Since H is EXNP-2-Absorbing submodule of M and $rs \notin [H + soc(M) + J(M) :_R M]$, it follows that either $re \in H + soc(M) + J(M)$ or $se \in H + soc(M) + J(M)$. Thus either $e \in [H + soc(M) + J(M) :_M r]$ or $e \in H + soc(M) + J(M) :_M s]$.Hence $e \in$

$[H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$. Therefore $[N :_M rs] \subseteq [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$.

($\Leftarrow$) Let $rse \in H$ for $r, s \in R$, $e \in M$ and let $rs \notin [H + soc(M) + J(M) :_R M]$. Then by our hypothesis $e \in [H :_M rs] \subseteq [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$. It follows that either $e \in [H + soc(M) + J(M) :_M r]$ or $e \in [H + soc(M) + J(M) :_M s]$. That is either $re \in H + soc(M) + J(M)$ or $\in H + soc(M) + J(M)$. Therefore H is EXNP-2-Absorbing submodule of M.

Proposition 4 A proper submodule H of M is EXNP-2-Absorbing submodule of M if and only if $rsB \subseteq H$, for $r, s \in R$ and B is a submodule of M, then either $rB \subseteq H + soc(M) + J(M)$ or $sB \subseteq H + soc(M) + J(M)$ or $rs \in [H + soc(M) + J(M) :_R M]$.

Proof

($\Rightarrow$) Let H be EXNP-2-Absorbing submodule of M and $rsB \subseteq H$, for $r, s \in R$ and B *is a* submodule of M. Let $rs \notin [H + soc(M) + J(M) :_R M]$, $rB \nsubseteq H + soc(M) + J(M)$ and $sB \nsubseteq H + soc(M) + J(M)$. Then there is $e_1, e_2 \in B$ in which $re_1 \notin H + soc(M) + J(M)$ and $se_2 \notin H + soc(M) + J(M)$. Now, $rse_1 \in H$ and $rs \notin [H + soc(M) + J(M) :_R M]$, then by Proposition 3 $e_1 \in [H :_M rs] \subseteq [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$, such that $e_1 \in [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$. But $re_1 \notin H + soc(M) + J(M)$, that is $e_1 \notin [H + soc(M) + J(M) :_M r]$. Thus $e_1 \in [H + soc(M) + J(M) :_M s]$, hence $se_1 \in H + soc(M) + J(M)$. Also since $rse_2 \in H$ and $rs \notin [H + soc(M) + J(M) :_R M]$ and $se_2 \notin H + soc(M) + J(M)$, it follows that $re_2 \in H + soc(M) + J(M)$. Now, $rs(e_1 + e_2) \in H$ and $rs \notin [H + soc(M) + J(M) :_R M]$, implies that $(e_1 + e_2) \in [H :_M rs]$. Next, following by Proposition 3 $(e_1 + e_2) \in [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$. That is either $r(e_1 + e_2) \in H + soc(M) + J(M)$ or $r(e_1 + e_2) \in H + soc(M) + J(M)$. If $r(e_1 + e_2) = re_1 + re_2 \in H + soc(M) + J(M)$ and $re_2 \in H + soc(M) + J(M)$, then $re_1 \in H + soc(M) + J(M)$ which is contradiction. If $s(e_1 + e_2) = se_1 + se_2 \in H + soc(M) + J(M)$ and $se_1 \in H + soc(M) + J(M)$, then $se_2 \in H + soc(M) + J(M)$ which is contradiction, hence either $rB \subseteq H + soc(M) + J(M)$ or $sB \subseteq H + soc(M) + J(M)$ or $rs \in [H + soc(M) + J(M):_R M]$.

($\Leftarrow$) Let $rs\mathfrak{m} \in H$ for $r, s \in R$, $\mathfrak{m} \in M$, then $rs(\mathfrak{m}) \subseteq H$, hence by hypothesis either $r(\mathfrak{m}) \subseteq H + soc(M) + J(M)$ or $s(\mathfrak{m}) \subseteq H + soc(M) + J(M)$ or $rs \in [H + soc(M) + J(M):_R M]$. That is either $r\mathfrak{m} \in H + soc(M) + J(M)$ or $s\mathfrak{m} \in H + soc(M) + J(M)$. Therefore H is EXNP-2-Absorbing submodule of M.

Proposition 5 Let M be module and H be a proper submodule of M. Then H is EXNP-2-Absorbing submodule of M if and only if for every submodule B of M and for every ideals I and $J$of R such that $IJB \subseteq H$ inplies that either $IB \subseteq H + soc(M) + J(M)$ or $JB \subseteq H + soc(M) + J(M)$ or $IJM \subseteq H + soc(M) + J(M)$.

Proof

($\Rightarrow$) Let $IJB \subseteq H$, where I, J are ideals of R and B is a submodule of M, with $IJ \nsubseteq [H + soc(M) + J(M):_R M]$. To demonstrate that $IB \subseteq H + soc(M) + J(M)$ or $JB \subseteq H + soc(M) + J(M)$. Suppose that $IB \nsubseteq H + soc(M) + J(M)$ and $JB \nsubseteq H + soc(M) + J(M)$, that is $there\ exist$ $c_1 \in I$ and $c_2 \in J$ such that $c_1B \nsubseteq H + soc(M) + J(M)$ and $c_2B \nsubseteq H + soc(M) + J(M)$. Now, $c_1c_2B \subseteq H$ and H is EXNP-2-Absorbing submodule of M, then by Proposition 3 either $c_1B \subseteq H + soc(M) + J(M)$ or $c_2B \subseteq H + soc(M) + J(M)$ or $c_1c_2 \in [H + soc(M) + J(M):_R M]$. Since $IJ \nsubseteq [H + soc(M) + J(M):_R M]$, then there exists $d_1 \in I$ and $d_2 \in J$ such that $d_1d_2 \notin [H + soc(M) + J(M):_R M]$. But, $d_1d_2B \subseteq H$ and H is EXNP-2-Absorbing submodule of M, and $d_1d_2 \notin [H + soc(M) + J(M):_R M]$, then by Proposition 3 either $d_1B \subseteq H + soc(M) + J(M)$ or $d_2B \subseteq H + soc(M) + J(M)$. Now, we have to discuss the following cases:

Case one: If $d_1B \subseteq H + soc(M) + J(M)$ and $d_2B \nsubseteq H + soc(M) + J(M)$. Since $c_1d_2B \subseteq H$ and $d_2B \nsubseteq H + soc(M) + J(M)$ and $c_1B \nsubseteq H + soc(M) + J(M)$, then by Proposition 3 $c_1d_2 \in [H + soc(M) + J(M):_R M]$. Since $d_1B \subseteq H + soc(M) + J(M)$ and $c_1B \nsubseteq H + soc(M) + J(M)$, we get $(c_1 + d_1)B \nsubseteq H + soc(M) + J(M)$. Moreover $(c_1 + d_1)d_2B \subseteq$

H, H is H is EXNP-2-Absorbing submodule of, $(c_1 + d_1)B \nsubseteq H + soc(M) + J(M)$ and $d_2B \nsubseteq H + soc(M) + J(M)$, by Proposition 3 $(c_1 + d_1)d_2 = c_1d_2 + d_1d_2 \in [H + soc(M) + J(M) :_R M]$. But $c_1d_2 \in [H + soc(M) + J(M) :_R M]$, next, following $d_1d_2 \in [H + soc(M) + J(M) :_R M]$, which is contradiction.

Case two: If $d_2B \subseteq H + soc(M) + J(M)$ and $d_1B \nsubseteq H + soc(M) + J(M)$, by similar case (1) we get a contradiction.

Case three: If $d_1B \subseteq H + soc(M) + J(M)$ and $d_2B \subseteq H + soc(M) + J(M)$, since $d_2B \subseteq H + soc(M) + J(M)$ and $c_2B \nsubseteq H + soc(M) + J(M)$, we get $(c_2 + d_2)B \nsubseteq H + soc(M) + J(M)$. But $(c_2 + d_2)c_1B \subseteq H$ and H is EXNP-2-Absorbing submodule of M with $c_1B \nsubseteq H + soc(M) + J(M)$ and $(c_2 + d_2)B\_\nsubseteq H + soc(M) + J(M)$ then by Proposition 3 we get $(c_2 + d_2)c_1 \in [H + soc(M) + J(M):_R M]$. Since $c_1c_2 \in [H + soc(M) + J(M):_R M]$ and $c_1c_2 + c_1d_2 \in [H + soc(M) + J(M):_R M]$, implies that $c_1d_2 \in [H + soc(M) + J(M):_R M]$. Now , since $(c_1 + d_1)c_2 \in H$ and $c_2B \nsubseteq H + soc(M) + J(M)$ and $(c_1 + d_1)B \nsubseteq H + soc(M) + J(M)$ and H is EXNP-2-Absorbing submodule of M, then by Proposition 3 we get $(c_1 + d_1)c_2 \in [H + soc(M) + J(M):_R M]$. But $(c_1 + d_1)c_2 = c_1c_2 + d_1c_2 \in [H + soc(M) + J(M):_R M]$, since $c_1c_2 \in [H + soc(M) + J(M):_R M]$, we get $d_1c_2 \in [H + soc(M) + J(M):_R M]$. Since $(c_1 + d_1)(c_2 + d_2)B \subseteq H$ and $(c_1 + d_1)B \nsubseteq H + soc(M) + J(M)$ and $(c_2 + d_2)B \nsubseteq H + soc(M) + J(M)$, then by Proposition 3 we have $(c_1 + d_1)(c_2 + d_2) = c_1c_2 + c_1d_2 + d_1c_2 + d_1d_2 \in [H + soc(M) + J(M):_R M]$. But c_1d_2, d_1c_2 , $c_1c_2 \in [H + soc(M) + J(M):_R M]$, so that $d_1d_2 \in [H + soc(M) + J(M):_R M]$ which is a contradiction. Consequently $IB \subseteq H + soc(M) + J(M)$ or $JB \subseteq H + soc(M) + J(M)$

(⇐) Suppose that $rsB \subseteq H$, where $r, s \in R$, B is a submodule of M then $(r)(s)B \subseteq H$, so by hypothesis, either $(r)B \subseteq H + soc(M) + J(M)$ or $(s)B \subseteq H + soc(M) + J(M)$ or $rsM \subseteq H + soc(M) + J(M)$. Hence either $rB \subseteq H + soc(M) + J(M)$ or $sB \subseteq H + soc(M) + J(M)$ or $rsM \subseteq H + soc(M) + J(M)$. Therefore by Proposition 3 get that H is EXNP-2-Absorbing submodule of M.

From Proposition 5, we get corollaries.

Corollary 6 Let M be a module and H be a proper submodule of M. Then H EXNP-2-Absorbing submodule of M if and only if for every submodule B of M, $r \in R$ and I ideal of R such that $rIB \subseteq H$ implies that either $rB \subseteq H + soc(M) + J(M)$ or $IB \subseteq H + soc(M) + J(M)$ or $rIM \subseteq H + soc(M) + J(M)$.

Corollary 7 Let M be a module and H be a proper submodule of M. Then H EXNP-2-Absorbing submodule of M if and only if $rIn \subseteq H, r \in R, n \in M$ and I ideal of R, then either $rn \subseteq H + soc(M) + J(M)$ or $In \subseteq H + soc(M) + J(M)$ or $rIM \subseteq H + soc(M) + J(M)$.

Lemma 8 [6, lemma (2.3.15)] Let$B$, K and D are submodules of an R-module M, with $\mathrm{K} \subseteq \mathrm{D}$, then $(B + \mathrm{K}) \cap \mathrm{D} = (B \cap \mathrm{D}) + \mathrm{K} = (B \cap \mathrm{D}) + (\mathrm{K} \cap \mathrm{D})$.

Proposition 9 Let H, B be EXNP-2-Absorbing submodules of M with B is not contained in H and either $soc(\mathrm{M}) + J(\mathrm{M}) \subseteq H$ or $soc(\mathrm{M}) + J(\mathrm{M}) \subseteq B$. Then $H \cap B$ EXNP-2-Absorbing submodule of M.

Proof

$H \cap B$ is a proper submodule of B and B is a proper submodule of M, hence $H \cap B$ is a proper submodule of M. Assume that $soc(\mathrm{M}) + J(\mathrm{M}) \subseteq B$ and $soc(\mathrm{M}) + J(\mathrm{M}) \nsubseteq H$ .Let $rsA \subseteq H \cap B$ for $r, s \in R$, A is a submodule of M, next that $rsA \subseteq H$ and $rsA \subseteq B$. But H, B are EXNP-2-Absorbing submodules of M, then either $rA \subseteq H + soc(\mathrm{M}) + J(\mathrm{M})$ or $sA \subseteq H + soc(\mathrm{M}) + J(\mathrm{M})$ or $rs\mathrm{M} \subseteq H + soc(\mathrm{M}) + J(\mathrm{M})$ and $rA \subseteq B + soc(\mathrm{M}) + J(\mathrm{M})$ or $sA \subseteq B + soc(\mathrm{M}) + J(\mathrm{M})$ or $rs\mathrm{M} \subseteq B + soc(\mathrm{M}) + J(\mathrm{M})$. Thus either $rA \subseteq (H + soc(\mathrm{M}) + J(\mathrm{M})) \cap (B + soc(\mathrm{M}) + J(\mathrm{M}))$ or $sA \subseteq (H + soc(\mathrm{M}) + J(\mathrm{M})) \cap (B + soc(\mathrm{M}) + J(\mathrm{M}))$ or $rs\mathrm{M} \subseteq (H + soc(\mathrm{M}) + J(\mathrm{M})) \cap (B + soc(\mathrm{M}) + J(\mathrm{M}))$. But $soc(\mathrm{M}) + J(\mathrm{M}) \subseteq B$, then $B + soc(\mathrm{M}) + J(\mathrm{M}) = B$, it follows that either $rA \subseteq (H + soc(\mathrm{M}) + J(\mathrm{M})) \cap B$ or $sA \subseteq (H + soc(\mathrm{M}) + J(\mathrm{M})) \cap B$ or $rs\mathrm{M} \subseteq (H + soc(\mathrm{M}) + J(\mathrm{M})) \cap B$. By Lemma 7 either $rA \subseteq$

$(H \cap B) +\ soc(\mathrm{M}) + J(\mathrm{M})$ or $sA \subseteq (H \cap B) +\ soc(\mathrm{M}) + J(\mathrm{M})$ or $rs\mathrm{M} \subseteq (H \cap B) +\ soc(\mathrm{M}) + J(\mathrm{M})$. Therefore $H \cap B$ is EXNP-2-Absorbing submodule of M.

Lemma 10 [7, remark, p 14] If M is a faithful multiplication R-module, then $J(\mathrm{M}) = \mathrm{J}(R)\mathrm{M}$.

Lemma 11 [8, Coro.(2.14) (i)] Let M be faithful multiplication R-module, then $soc(\mathrm{M}) = soc(R)\mathrm{M}$.

Proposition 12 Let K be a proper submodule of faithful multiplication R-module. Then K is EXNP-2-Absorbing submodule of M if and only if $[K:_R M]$ is EXNP-2-Absorbing ideal of R.

Proof

(⟹) Assume that K is EXNP-2-Absorbing submodule of M, and $rst \in [K:_R M]$ for $r,s,t \in R$, then $rs(tM) \subseteq K$. But K is EXNP-2-Absorbing submodule of M, then by Corollary 6 either $r(tM) \subseteq K + soc(M) + J(M)$ or $s(tM) \subseteq K + soc(M) + J(M)$ or $rsM \subseteq K + soc(M) + J(M)$. Since M is multiplication, then $K = [K:_R M]M$, and since M is faithful multiplication, then by Lemma 11 $soc(M) = soc(R)M$ and by Lemma 10 $J(M) = J(R)M$. Thus either $r(tM) \subseteq [K:_R M]M + soc(R)M + J(R)M$ or $s(tM) \subseteq [K:_R M]M + soc(R)M + J(R)M$ or $rsM \subseteq [K:_R M]M + soc(R)M + J(R)M$. Hence either $rt \in [K:_R M] + soc(R) + J(R)$ or $st \in [K:_R M] + soc(R) + J(R)$ or $rs \in [K:_R M] + soc(R) + J(R) = \big[[K:_R M] + soc(R) + J(R): R\big]$. Therefore $[K:_R M]$ is EXNP-2-Absorbing ideal of R.

(⟸) Suppose that $[K:_R M]$ is EXNP-2-Absorbing ideal of R, and $rsB \subseteq K$ for $r,s \in R$, B is a submodule of M. Since M is a multiplication, then $B = IW$ for some ideal I of R, that is $rsIM \subseteq K$, hence $rsI \subseteq [K:_R M]$, but $[K:_R M]$ is EXNP-2-Absorbing ideal of R, then either $rI \subseteq [K:_R W] + soc(R) + J(R)$ or $sI \subseteq [K:_R M] + soc(R) + J(R)$ or $rs \in \big[[K:_R M] + soc(R) + J(R):_R R\big] = [K:_R M] + soc(R) + J(R)$. Hence either $rIW \subseteq [K:_R M]M + soc(R)M + J(R)M$ or $sIM \subseteq [K:_R M]M + soc(R)M + J(R)M$ or $rsM \subseteq [K:_R M]M + soc(R)M + J(R)M$. That is either $rB \subseteq K + soc(M) + J(M)$ or $sB \subseteq K + soc(M) + J(M)$ or $rs \in [K + soc(M) + J(M):_R M]$. Thus K is EXNP-2-Absorbing submodule of M.

Lemma 13 [9] Let M be a multiplication finitely generated module $\mathcal{A}, \mathcal{B}$ are ideals $of\ R$, then $\mathcal{A}M \subseteq \mathcal{B}M$ if and only if $\mathcal{A} \subseteq \mathcal{B} + ann(M)$.

Proposition 14 Let M be a faithful finitely generated multiplication module and $\mathcal{A}$ is ideal of R. Then $\mathcal{A}$ is EXNP-2-Absorbing ideal of R if and only if $\mathcal{A}M$ is EXNP-2-Absorbing submodule of M.

Proof

($\Longrightarrow$) Let $rsB \subseteq \mathcal{A}M$ for any $r, s \in R$, B is a submodule of M. Since M is a multiplication, then $B = IM$ for some ideal $I\ of\ R$, that is $rsIM \subseteq \mathcal{A}M$. Thus by lemma 13 get $rsI \subseteq \mathcal{A} + ann(M)$, but M is faithful, it follows $ann(M) = \{0\}$, that is $rsI \subseteq \mathcal{A}$. Since $\mathcal{A}$ is EXNP-2-Absorbing ideal of R, then by Proposition 4 either $rI \subseteq \mathcal{A} + soc(R) + J(R)$ or $sI \subseteq \mathcal{A} + soc(R) + J(R)$ or $rs \in [\mathcal{A} + soc(R) + J(R) :_R R] = \mathcal{A} + soc(R) + J(R)$. Hence either $rIM \subseteq \mathcal{A}M + soc(R)M + J(R)M$ or $sIM \subseteq \mathcal{A}M + soc(R)M + J(R)M$ or $rsM \subseteq \mathcal{A}M + soc(R)M + J(R)M$, so that by Lemma 10 and Lemma 11 either $rB \subseteq \mathcal{A}M + soc(M) + J(M)$ or $sB \subseteq \mathcal{A}M + soc(M) + J(M)$ or $rsM \subseteq \mathcal{A}M + soc(M) + J(M)$. Thus $\mathcal{A}M$ is an EXNP-2-Absorbing submodule of M.

($\Longleftarrow$) Let $rsI \subseteq \mathcal{A}$ for $r, s \in R$ and I ideal of R, hence $rs(IM) \subseteq \mathcal{A}M$, but $\mathcal{A}M$ is an EXNP-2-Absorbing submodule of M, then either $r(IM) \subseteq \mathcal{A}M + soc(M) + J(M)$ or $s(IM) \subseteq \mathcal{A}M + soc(M) + J(M)$ or $rsM \subseteq \mathcal{A}M + soc(M) + J(M)$. Thus by Lemma 10 and Lemma 11 either $rIM \subseteq \mathcal{A}M + soc(R)M + J(R)M$ or $sIM \subseteq \mathcal{A}M + soc(R)M + J(R)M$ or $rsM \subseteq \mathcal{A}M + soc(R)M + J(R)M$, hence either $rI \subseteq \mathcal{A} + soc(R) + J(R)$ or $sI \subseteq \mathcal{A} + soc(R) + J(R)$or $rs \in \mathcal{A} + soc(R) + J(R) = [\mathcal{A} + soc(R) + J(R) :_R R]$. Therefore $\mathcal{A}$ is EXNP-2-Absorbing ideal of R.

Conclusion

In this paper, we introduced a new concept, which is an Extend Nearly Pseudo-Absorbing submodule as a generalization of 2-absorbing submodules. The following are some of the most important outcomes of this work.

1) Every (2-absorbing , nearly-2-absorbing and pseudo-2-absorbing) submodules of M is EXNP-2-Absorbing submodule, but the converse is not true in general.

2) A proper submodule H of M is EXNP-2-Absorbing submodule of M if and only if for any $r, s \in \Re$ such that $rs \notin [H + soc(M) + J(M) :_R M]$ we have $[H :_M rs] \subseteq [H + soc(M) + J(M) :_M r] \cup [H + soc(M) + J(M) :_M s]$.

3) Let K be a proper submodule of faithful multiplication R-module. Then K is EXNP-2-Absorbing submodule of M if and only if $[K :_R M]$ is EXNP-2-Absorbing ideal of R.

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