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On Nano β_{pc}-Open Set and Some of Its Application

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Abstract

The major objective of this work is to establish and assemble a new type of Nano β-open termed a Nano-β_{PC} open set, which is an advanced form of Nano β-open. Additionally, we identify a class of Nano β_{PC}-open sets in terms of Nano topology elements. Furthermore, it has a number of given properties. In this paper, we introduce a completely novel kind of Nano continuity that we've labeled Nano β_{PC}-continuous functions, and we address many of its characterizations.

Key words: Nano-β_{PC} open set, Nano-β_{PC} closed set, Nano pre-regular, Nano β_{PC}- continuous function, Nano pre-θ open set.

Introduction

Rough set theory was developed by Pawalak [1] and used as a tool to describe thinking and decision-making. By establishing the bounds and approximations, he looked at several key components of this theory. In addition to the boundary region of a subset of some universe using equivalent relation upon it, Thivagar and colleagues [2] offered Nano topology described in aspects of lower as well as higher approximations. The definitions of (semi, pre, and)-open set were later published by Thivagar & Richard [3]. They also talked about the fundamental characteristics of set closures and interiors at the nanoscale. Ibrahim [4] provided significant descriptions of the operation of a collection of Nano-open sets., Parimala and others[5] researched nanostructures with b-open sets and used this word to explore a novel kind of continuity called a Nano continuous function. The phrase "Nano Z-open set" was also proposed by Selvaraj and Balakrishna[6], who used it to characterize the fundamental and crucial characteristics of this kind of set. This set was compared to the weak and strong versions of the Nano open sets. Vadivel and others [7] were interested to discuss the properties and preliminary results of the Nano θ-open set as well as how to apply the concept to the medical field. Jamil and Wadhah [8] presented on a novel class of Nano open sets termed Nano ιc-open, and examined the relationships between various variants and this set. Pinpointed the primary component causing anemia as a real-world application of Nano topology. In this work, a new Nano open set have been developed namely Nano-β_{PC} open set and employ this notion to present a new type of continuous function is named Nano-β_{PC} continuous function such that several characterizations are investigated.

1. Pre-information's

Definition 1.1 [3] considering $\mathcal{V}$ be universe, comprise of limited many things and RL_{eq} be an equivalence relation on RL_{eq} recognize as the indiscernibility relation. Consequently $\mathcal{V}$ is partitioned to equivalence classes which are disjoint indistinguishable elements are those that belong to a certain equivalence class. A pairing $(\mathcal{V}, RL_{eq})$ is named approximation space .Letting $D \subseteq \mathcal{V}$

G1- The whole assortment of items with labels D with respect to $\mathfrak{R}$ is the lower approximation of X w.r.t. RL_{eq} , which is symbolized by $L_{RL_{eq}}(D)$. That is $L_{RL_{eq}}(D) = \bigcup_{x\in\mathcal{V}}\{RL_{eq}(x): RL_{eq}(x) \subseteq D\}$ such that $RL_{eq}(x)$ is represented the set of the equivalence classes determined by$x$.

G2- The whole assortment of items with possibly tags as D regard to $\mathfrak{R}_q$ is namely upper approximation of X regarding to R_q. Equivalently, $U_{RL_{eq}}(D) = \bigcup_{x\in\mathcal{V}}\{RL_{eq}(x): RL_{eq}(x) \cap D \neq \phi\}$

G3- The set of any things can be categorized not D nor as non X with respect to RL_{eq} is known as the boundary region of X regards toR_q. Equivalently, $B_{R_q}(D) = U_{R_q}(D) - L_{R_q}(D)$.

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Definition 1.2 [3] Letting $\mathcal{V}$ stand for the universe and R_q stand for equivalence relation on $\mathcal{V}$ equipped with $\tau_{R_q}(X) = \left\{\mathcal{V}, \phi, U_{RL_{eq}}(D), B_{RL_{eq}}(D), L_{RL_{eq}}(D)\right\}$ that is D is subset of $\mathcal{V}$ and $\tau_{RL_{eq}}(X)$ addressed the below conditions

T1- $\mathcal{V}$ and ϕ are elements of $\tau_{RL_{eq}}(D)$

T2- the union of all elements of subfamily of $\tau_{RL_{eq}}(D)$ is also, in$\tau_{RL_{eq}}(D)$.

T3- the intersection of finitely many subfamily of $\tau_{RL_{eq}}(D)$ is also, in$RL_{eq}(D)$.

Then the family $\tau_{RL_{eq}}(D)$ is known as Nano topology on $\mathcal{V}$ regarding to D and the pair $\left(\mathcal{V}, \tau_{RL_{eq}}(D)\right)$ is named a Nano topological space. N Top S is our abbreviation.

The members of $\tau_{R_q}(D)$ are named as Nano open set. Also, the Nano open set's complement knows as Nano closed set. Nano open and Nano closed sets are abbreviated as N-open and N-closed respectively.

Definition 1.3 [3] For a $\mathfrak{N}$Top S $\left(\mathcal{V}, \tau_{RL_{eq}}(X)\right)$ regarding to D where$D \subseteq \mathcal{V}$. If L is a subset of $\mathcal{V}$, the Nano interior of L is the union of every Nano open sets contained in L as well as its symbolized by $\mathfrak{N}int(L)$ and the Nano closure of W is the intersection of all Nano closed sets containing W as well as its symbolized by $\mathfrak{N}clo(L)$.

Remark 1.4 [3] The family $\beta = \left\{\mathcal{M}, L_{RL_{eq}}(X), B_{RL_{eq}}(X)\right\}$ forms a base for a Nano topology $\tau_{RL_{eq}}(X)$ on the universal set$\mathcal{M}$.

Definition 1.5 [3] A subset H of $\mathfrak{N}$Top S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is named

S1- Nano semi-open if $H \subseteq \mathfrak{N}clo\ \mathfrak{N}int(H)$

S2- Nano pre-open if $H \subseteq \mathfrak{N}int\ \mathfrak{N}clo(H)$

S3- Nano α-open if $H \subseteq\ \mathfrak{N}int\ \mathfrak{N}clo\ \mathfrak{N}int(H)$

S4- Nano β-open if $H \subseteq \mathfrak{N}clo\ \mathfrak{N}int\ \mathfrak{N}clo(H)$

The proof following proposition is routine, it is omitted.

Remark 1.6 Every Nano open set is Nano pre - open. However the converse may not be true, as showing in the next example.

Example 1.7 Let $M = \{ a, b, c, d \}\, with\, {}^{M}/_{R} = \{ \{a\}, \{b\}, \{c, d\} \}, set\ X = \{ a, c, \}$ .Then $\tau_R(X) = \{M, \phi, \{a\}, \{c, d\}, \{a, c, d\}\}$ and family of Nano closed is $\Gamma = \{\phi, M, \{b\}, \{a, b\}, \{b, c, d\}\}$, then $K = \{a, c\}$ is Nano pre – open but it's not Nano open.

Proposition 1.8

(S1) Every Nano α –open is Nano pre – open.

(S2) Every Nano pre – open is Nano $-open$.

$$Nano\ open \longrightarrow Nano\ \alpha - open \longrightarrow Nano\ pre - open \longrightarrow Nano\ \beta - open$$

Proposition 1.9 The union of Nano β – open sets are also Nano β – open.

Remark 1.10 The intersection of two Nano β – open sets may not be a Nano β – open set.

Example 1.11 Consider the same Example in 2.7, $M = \{ a, b, c, d\}$ with ${}^{M}/_{R} = \{ \{a\}, \{b\}, \{c, d\} \}$, $X = \{a, c\}$. Then $\tau_R(X) = \{ \emptyset, M, \{a\}, \{c, d\}, \{a, c, d\}\}$ and $\tau_R(X)$ – closed set are $\emptyset, M, \{b\}, \{a, b\}, \{b, c, d\}$. Let $K = \{a, b\}$ and $H = \{ b, c\}$ .Then K and H are both Nano β – open while $K \cap H = \{b\}$ is not Nano β – open set.

Remark 1.12 The family of Nano β – open sets are Supra topology.

A sub-collection φ of the power set $P(X)$ is named a supra topology on X if φ satisfies the following conditions.

1) $\phi, X \in \varphi$
2) φ is closed under the arbitrary union.

Proposition 1.13 if V is Nano open and L is Nano β - open subset of $\mathfrak{N}$.Top .S$\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ .Then $V \cap L$ is Nano β - open.

Definition 1.14 A $\mathfrak{N}$.Top .S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is said to be Nano pre T_1-space if every two distinct points s , t there are two Nano pre -open sets $K$and L such that $s \in K, t \notin K$ and $t \in L, s \notin L$.

Proposition 1.15 A $\mathfrak{N}$.Top .S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is Nano pre T_1 -space if and only if every singleton is Nano pre– closed

Definition 1.16 Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$ be two $\mathfrak{N}$.Top.S,A function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$ is called Nano β – continuous if the inverse image of every Nano – open set in N is Nano β – open in M

Definition 1.17 Let $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ and $\left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$ be two $\mathfrak{N}$. Top. S, a function $f: \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$ is named β -Irresolute, if $f^{-1}(\mathcal{B}) \in \mathfrak{N}\beta O\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ for every $\mathcal{B} \in \mathfrak{N}\beta O\left(\mathcal{H}, \tau_{RL^*{}_{eq}}(X)\right)$

2. Nano-β_{PC} open Set

Definition 2.1 A Nano β-open subset B of $\mathfrak{N}$.Top .S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is named Nano β_{PC}-open if for each $\in B$, there is Nano pre-closed F such that $x \in F \subseteq B$. The family of all β_{PC}-open sets is symbol led by $\mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$

Definition 2.2 A Nano subset H of $\mathfrak{N}$.Top.S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ is named β_{PC}-closed if it`s complement is Nano β_{PC}-open. The family of all β_{PC}-open sets is symbolled by $\mathfrak{N}\beta_{PC}c$ $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$

Remark 2.3 Every Nano β_{PC}-open set is Nano β -open but the converse may not be true as showing in the following example.

Example 2.4 Let $M = \{a, b, c, d\}$ with ${}^{M}/_{R} = \{\{a\}, \{b\}, \{c, d\}\}$ set $X = \{a, c\}$ then $\tau_R(x) = \{\phi, M, \{a\}, \{c, d\}, \{a, c, d\}\}$ clearly, the family τ_R closed is $\varphi = \{\phi, M, \{b\}, \{a, b\}, \{b, c, d\}\}$ and the family of $\mathfrak{N}$ β o $\left(M, T_R(x)\right) = \{\phi, M, \{a\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, d\}\{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. Not that $K = \{d\}$ is Nano β-open but it is not β_{PC}-open set.

Remark 2.5 Nano β_{PC}-open set and Nano open are independent. In previous example, $\{a\}$ is Nano open but it is not Nano β_{PC}-open while $\{a\ ,b\ ,d\}$ is Nano β_{PC}-open but it is not Nano open.

Definition 2.6 A Nano subset of $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is named Nano pre - Regular set if it's both Nano pre-open and Nano pre-closed.

Proposition 2.7 Every Nano pre- Regular set is Nano β_{PC}-open.

But the converse may not be true as showing in the following example.

Example 2.8 Let $\mathcal{M} = \{\ c,d,f\}$ with ${}^{M}/_{R} = \{\ \{c\},\{\ d,f\ \}\ \}$. Set $X = \{d\ ,f\ \}$ then $\tau_R(X) = \{\phi,\mathcal{M},\{\ d,f\ \}\}$ consequently $\mathfrak{N}\,\beta\,\mathrm{o}\left(\ \mathcal{M}\ ,\tau_R(X)\right) = \{\phi\ ,M\ ,\{d\}\ ,\{f\},\{\ c,d\ \},\{\ c\ ,f\ \}\ ,\{\ d,f\ \}\}$ and $\mathfrak{N}\mathcal{P}\mathrm{c}(\mathcal{M},X) = \{\phi,\mathcal{M},\{c\},\{d\},\{f\},\{c,f\}\ ,\{c,d\}\}$. Then $\{d,f\}$ is Nano β_{PC}-open but it is not Nano pre-Regular.

Proposition 2.9 Let S_i be Nano β_{PC}-open sets in $\mathfrak{N}$.Top.S$\left(\mathcal{M}\ ,\tau_R(X)\right)$ for each i. Then $\bigcup_{i=1}^{n} S_i$ is Nano β_{PC}-open set.

Proof: Since S_i is Nano β_{PC}-open for every. Then by Proposition 1.3. $\bigcup_{i=1}^{n} S_i$ is Nano β -open. Assume $p\ \in \bigcup_{i=1}^{n} S_i$, then there is Nano pre-closed set $F_{\lambda 0}$ such that $p \in F_{\lambda 0}\ \subseteq \bigcup_{i=1}^{n} S_i$ that is $p \in F_{\lambda 0}\ \subseteq \bigcup_{i=1}^{n} S_i$. Hence $\bigcup_{i=1}^{n} S_i$ is Nano β_{PC}-open set.

Remark 2.10 The intersection of two Nano β_{PC}-open may not be a Nano β_{PC}-open. As showing in the example.

Example 2.11 Let $\mathcal{M} = \{a\ ,b,c,d\ \}$, ${}^{\mathcal{M}}/_{R} = \{\ \{a,b\},\{c\},\{d\}\}\ and\ X = \{a,c\}$ thus $\tau_R(X) = \{\emptyset,M,\{a,b,c\},\{c\},\{a,b\}\}$. Accordingly, $\mathfrak{N}\ \beta\ \mathrm{o}\ \left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \{\{a\},\{b\},\{c\},\{a,c\},\{a,b\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$ and $\mathfrak{N}\mathcal{P}\mathrm{c}\ \left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \{\{a\},\{b\},\{d\},\{a,d\},\{c,d\},\{b,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$. The family $\mathfrak{N}\,P_{PC}\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \{\{a\},\{b\},\{a,b\},\{a,c,d\},\{b,c,d\}\}$. The set $K = \{c,d\}$ and $= \{b,d\}$, so $K \cap H = \{d\}$ is not Nano β_{PC}-open set .

Remark 2.12 The class of Nano β_{PC}-open sets is supra topology.

Definition 2.13 For a Nano subset H of $\mathfrak{N}$..Top.S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is Nano $P\theta$-open set if for each $x \in H$, there exists a Nano open set G such that $x \in G \subseteq \mathfrak{N}PclO(G) \subset H$.

Proposition 2.14 Every Nano $P\theta$-open set is Nano β_{PC}-open.

Proof: Obvious.

Proposition 2.15 If a $\mathfrak{N}$. Top .S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is Nano pre T_1 - space, then $\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \mathfrak{N}\beta o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$

Proof: Let K be any subset of $\mathfrak{N}$.Top .S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ and$K \in \mathfrak{N}\mathcal{P}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. If $K = \phi$, then $K \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. Otherwise$K \neq \phi$, consider$p \in K$, and since $\mathfrak{N}$.Top .S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is Nano pre T_1- space, then by Proposition 2.15, $\{p\} \in \mathfrak{N}\mathcal{P}c\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. It follows that $p \in \{p\} \subseteq K$, so $K \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. Consequently, $\mathfrak{N}\beta o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) \subseteq \mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ and since $\mathfrak{N}\beta o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) \subseteq \mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. Hence $\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \mathfrak{N}\beta o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$.

Proposition 2.16 If $L_{Rq}(X) = \phi$, then every subset of $\mathfrak{N}$.Top .S. $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ is Nano pre - open set (resp., Nanoβ_{PC}-open set)

Proof: Since $L_R(X) = \phi$, Then $\tau_R(X)$ consists of M , $\phi, and\ U_R(X)$, if $K \subset U_R(X)$, then the only Nano closed set containing $K$is M . Consequently, $\left(\mathfrak{N}int\left(\mathfrak{N}clo(K)\right)\right) = \left(\mathfrak{N}int(M)\right) = M$. so K is Nano-open set. If $U_R(X) \subset K$, $\mathfrak{N}clo\left(\mathfrak{N}int\left(\mathfrak{N}clo(K)\right)\right) = M$,then K is also Nano pre– open. It follows that $\mathfrak{N}\mathcal{P}c\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \mathfrak{N}P(M)$. Accordingly, K is Nano pre-regular. By Proposition 2.9, K is Nanoβ_{PC}-open set. Hence $\mathfrak{N}\beta_{PC}o\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) = \mathfrak{N}\mathcal{P}(M)$

Proposition 2.17 If $U_R(X) = M\ and\ L_R(X) \neq \phi$ in $\mathfrak{N}$.Top .S $\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$. Then every subset of $\mathcal{M}$ is Nano pre– open (resp., Nanoβ_{PC}-open set).

Proof : Since $U_R(X) = M\ and\ L_R(X) \neq \phi$, then $\tau_R(X) = \{ \phi, M , L_R(X) , [L_R(X)]^{c}\}$. If $\subset L_R(X)$, then $\mathfrak{N}int\ (\mathfrak{N}cl(K) = \mathfrak{N}int(L_R(X)) = L_R(X)$. That is $K \subset L_R(X) = \mathfrak{N}int(\mathfrak{N}clo(K))$. Hence K is Nano pre –open. If $L_R(X) \subset K$, then $\mathfrak{N}int\ (\mathfrak{N}clo(K)) = \mathfrak{N}int(M) = M$. Hence K is Nano pre –open. The same argument for $L^{C}{}_R(X)$, that is every subset of M is Nano pre – open. Accordingly, K is Nano pre-regular. By Proposition 2.9, K is Nano β_{PC}-open set. Hence $\mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) = \mathfrak{N}P(M)$

Proposition 2.18 Let $U_R(X) = L_R(X) = X$ in $\mathfrak{N}$. Top .S $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$ Then $\mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) = \mathfrak{N}P(M)$ (The set of all subset of M).

Proof : Since $U_R(X) = L_R(X) = X$, then $\tau_R(X) = \{\phi, M , X\}$ and the family of Nano closed consists$\phi, M , M - X$. If $K \subset X \subset M$, then the only Nano closed containing K is M , so $\mathfrak{N}int\ (\mathfrak{N}\ cl(K) = \mathfrak{N}int(M) = M$ accordingly , K is Nano pre – open . If $\subset K \subset M$, then the only Nano closed containing K is also M , therefore $\mathfrak{N}int\ (\mathfrak{N}cl(K) = \mathfrak{N}int(M) = M$, that is every subset of M is Nano pre – open. Accordingly, K is Nano pre-regular. By Proposition 2.9, K is Nano P_{PC}-open set. Hence $\mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) = \mathfrak{N}P(M)$

Proposition 2.19 Let $L_R(X) \neq U_R(X)\ where\ L_R(X) \neq \phi\ and\ U_R(X) \neq \phi$.

(W1) If $A \subset L_R(X)$, then $A \in \mathfrak{N}P_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$

Proof: Since $A \subset L_R(X)$, then $\mathfrak{N}clo\,\mathfrak{N}int\,\mathfrak{N}clo(A) = \mathfrak{N}clo\,\mathfrak{N}int\ (\ \mathcal{M} - B_{Rq}(X))$. But $\mathfrak{N}clo\left(\mathcal{M} - U_{Rq}(X)\right) = \mathfrak{N}clo\left(\mathcal{M} - \left(U_{Rq}(X) \cap L^{c}{}_{Rq}(X)\right)\right)$

$= \mathfrak{N}clo([\mathcal{M} - U_{Rq}(X)] \cup L_{Rq}(X)) = \mathcal{M} - B_{Rq}(X) = ([\mathcal{M} - U_{Rq}(X)] \cup L_{Rq}(X))$, therefore the largest Nano open subset of $\mathcal{M} - U_R(X) \cup L_R(X)$ must containing $L_R(X)$ which is a superset of A. That is A is Nano β– open.

On other hand $\mathfrak{N}int\ (A) = \mathfrak{N}clo(\phi) = \phi$. Therefore A is Nano pre –closed. Hence $A \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$.

(W2) If $A \subset B_R(X)$, Then its $\mathfrak{N}clo\ \mathfrak{N}int\ \mathfrak{N}clo(A)$= $\mathfrak{N}clo\ \mathfrak{N}int\ ([\mathcal{M} - U_{Rq}(x)] \cup B_{Rq}(X)) = \mathfrak{N}clo\left(\mathcal{M} - L_R(X)\right)$. Consequently $A \subseteq \mathfrak{N}clo\ \mathfrak{N}int\ \mathfrak{N}clo(A)$ therefore A is Nano pre – open. Also$t\ \mathfrak{N}clo\ \mathfrak{N}int\ (A) = \mathfrak{N}clo(\phi) = \phi \subseteq A$. That is A is Nano pre – closed .Accordingly, $A \in \mathfrak{N}\beta_{PC}o\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$

Definition 2.20 A point $p \in M$ is known as $\mathfrak{N}\beta_{PC}$ – interior point of $K \subseteq M$, if there is a $\mathfrak{N}\beta_{PC} - open$ L ($L \in \mathfrak{N}P_{Pc}o\left(M, \tau_R(X)\right)$ having p such that $\in L$. The set of all $\mathfrak{N}\beta_{PC} - interior$ point of K is named $\mathfrak{N}\beta_{PC} - interior$ and symbolled by $P_{Pc} - int(K)$.

Theorem 2.21 Let K be a subset of $\mathfrak{N}$.Top.S. $\left(M, \tau_R(X)\right)$. Then

D1) $\mathfrak{N}\beta_{PC}int(K) \subseteq K$

D2) $K \in \mathfrak{N}\beta_{PC}o\left(M, \tau_R(X)\right)$ if and only if $K = \mathfrak{N}\beta_{PC}int(K)$

D3) $\mathfrak{N}\beta_{PC}int(\phi) = \phi\ and\ \mathfrak{N}\beta_{PC}int(M) = M$

D4) if $A \subseteq B$ $, then\ \mathfrak{N}\beta_{PC}int(A) \subseteq \mathfrak{N}\beta_{PC}int(B)$

D5) $\mathfrak{N}\beta_{PC}int(A) \cup \mathfrak{N}\beta_{PC}int(B) \subseteq \mathfrak{N}\beta_{PC}int(A \cup B)$

D6) $\mathfrak{N}\beta_{PC}int(A \cap B) \subseteq \mathfrak{N}\beta_{PC} - int(A) \cap \mathfrak{N}\beta_{PC} - int(B)$.

Proof: obvious.

Definition 2.22 Let A be a subset of $\mathfrak{N}$.Top.S. $\left(\mathcal{M}, \tau_R(X)\right)$ is said to be $\mathfrak{N}\beta_{PC} - closure$ of A define as the intersection of all $\mathfrak{N}\beta_{PC} - closed$ containing A , that means $\mathfrak{N}\beta_{PC}clo(A) = \cap \left\{K \supseteq A : K \in \mathfrak{N}\beta_{PC}C\left(\mathcal{M}, \tau_R(X)\right)\right\}$.

Theorem 2.23 Let K be any subset of $\mathfrak{N}$.Top.S. $\left(\mathcal{M}, \tau_R(X)\right)$ And let $p \in J$, then $p \in \mathfrak{N}\beta_{PC}clo(K)$ if and only if $K \cap J \neq \emptyset$ for every Nano β_{PC} – open J with $\in J$.

Proposition 2.24 For any subset A and B of $\mathfrak{N}$.Top.S. $\left(\mathcal{M}, \tau_R(X)\right)$, then

W1) If $A \subseteq B$ $, then\ \mathfrak{N}\beta_{PC}clo(A) \subseteq \mathfrak{N}\beta_{PC}clo(B)$

W2) $\mathfrak{N}\beta_{PC}clo(A \cap B) \subseteq \mathfrak{N}\beta_{PC}clo(A) \cap \mathfrak{N}\beta_{PC}clo(B)$

W3) $\mathfrak{N}\beta_{PC}clo(A) \cup \mathfrak{N}\beta_{PC}clo(B) \subseteq \mathfrak{N}\beta_{PC}clo(A \cup B)$

W4) $K \subseteq \mathfrak{N}\beta_{PC}clo(K)$

W5) $\beta_{PC}clo(\phi) = \phi\ and\ \mathfrak{N}\beta_{PC}clo(M) = M$.

Proof: Obvious.

Example 2.25 Let $\mathcal{M} = \{ a,b,c,d\}, {}^{\mathcal{M}}/_{R} = \{ \{a,b\},\{c\},\{d\}\}\ and\ X = \{ a,c\}$, then $\tau_R(X) = \{ \mathcal{M}, \phi, \{c\},\{a,b\},\{a,b,c\}\} . Accordigly\ \mathfrak{N}\beta\text{o}\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) = \{\{a\},\{b\},\{c\},\{a,c\},\{a,b\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$. Also, $\mathfrak{N}\beta c(\mathcal{M} , \tau_R(X)) = \{\{a\},\{b\},\{d\},\{a,d\},\{b,d\},\{c,d\},\phi,\mathcal{M},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$ and $\mathfrak{N}\beta_{PC}O(\mathcal{M} , \tau_R(X)) = \{\phi,\mathcal{M},\{a\},\{b\},\{a,b\},\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$. Let $K = \{a,d\} , J = \{c,d\}. Then\ \{d\} = \{a,d\} \cap \{c,d\} = \mathfrak{N}\beta_{PC} - int(K) \cap \mathfrak{N}\beta_{PC} - int(J) \nsubseteq \mathfrak{N}\beta_{PC} - int(K \cap J) = \mathfrak{N}\beta_{PC} - int(\{d\}) = \phi$. Also, let $S = \{ c\}\ and\ T = \{d\}$ then $\{c,d\} = \mathfrak{N}\beta_{PC} - int(\{c\} \cup \{d\}) \nsubseteq \mathfrak{N}\beta_{PC} - int(\{c\}) \cup \mathfrak{N}\beta_{PC} - int(\{d\} = \phi \cup \phi = \phi$. also, the family of Nano β_{PC} – closed set consists $\phi, M, \{a\},\{b\},\{c\},\{c,d\},\{b,c\} , \{a,c\} , \{a,b\} , \{b,c,d\} , \{a,c,d\}$. Let $K = \{c\} , J = \{d\}$, then $\mathfrak{N}\beta_{PC}clo(K) \cap \mathfrak{N}\beta_{PC}clo(J) = \{c\} \cap \{c,d\} = \{c\} \nsubseteq \phi = \mathfrak{N}\beta_{PC}clo(K \cap J)$.

3. Nano-β_{PC} Continuous functions

Definition 3.1 A function $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is known as Nano β_{PC} – continuous, if $f^{-1}(K)$ in M is Nano β_{PC} – open for every Nano β – open set K in $\mathfrak{N}$

Proposition 3.2 For a function $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$,then f is Nano β_{PC}– continuous if and only if for each $p \in M$ and any Nano β – open $L\ in\ N$,with $f(p) \in L$, then there exists Nano β_{PC} – open set H in M with $f(H) \subseteq L$.

Proof: Suppose that f is Nano β_{PC} – continuous function and let $p \in M$ and let L be any Nano pre – open containing$f(p)$, it follows $p \in f^{-1}(L)$ by hypothesis, $f^{-1}(L) = H$ is Nano β_{PC} – open set M and $f(H) = f\left(f^{-1}(L)\right) \subseteq L$.

Conversely, let L be a Nano β – open set in $\mathfrak{N}$.Top.S. $(N, \tau^*{}_R(Y))$, if $f^{-1}(L) = \phi$, then $f^{-1}(L)$ is Nano β_{PC} – open , if $f^{-1}(L) \neq \phi$, then there is $p \in f^{-1}(L)$ consequently $f(p) \in L\ and\ since\ L$ is Nano β – open , then by hypothesis there exists Nano β_{PC}– open set H_p in M with $f(H) \subseteq L$ for every $p \in H$. It follows $\in H \subseteq f^{-1}(L)$. Hence $f^{-1}(L)$ is Nano β_{PC} – open set.

Proposition 3.3 A function $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ is β_{PC} – continuous if and only if $f^{-1}(K)$ is Nano $\beta_{PC} - closed\ in\ \mathcal{M}$ for every Nano β – closed K in .

Proof: Obvious.

Proposition 3.4 For a function $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ the following characterizations are equivalent

E1) $f\ is\ \beta_{PC}$ −continuous.

E2) $f\left(\mathfrak{N}\beta_{PC}clo(K)\right) \subseteq \mathfrak{N}\beta clo(\ f(K))\ , for\ K\ in\ M$

E3) $\mathfrak{N}\beta_{PC}clo(\ f^{-1}(K)) \subseteq f^{-1}\left(\mathfrak{N}\beta clo(K)\right) for\ every\ K\ in\ N$.

Proof: E1 ⇒ E2 .Let K be a subset of $\mathfrak{N}$.Top.S. $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then $\mathfrak{N}\beta cl\left(F(K)\right)$ is Nano β – closed in $\mathfrak{N}$, since f is Nano β_{PC} – continuous , then by Proposition 4.3 , $f^{-1}\left(\mathfrak{N}\beta clo\left(f(K)\right)\right)$ is Nano β_{PC} − closed in M and since $K \subseteq f^{-1}\left(\mathfrak{N}\beta cl\left(f(K)\right)\right)$, then $\mathfrak{N}\beta_{PC}cl(K) \subseteq f^{-1}\left(\mathfrak{N}\beta cl\left(f(K)\right)\right)$ so , $f\left(\mathfrak{N}\beta_{PC}clo(K)\right) \subseteq \mathfrak{N}\beta cl\left(f(K)\right)$.

E2 ⇒ E3 . Let K be any subset of $\mathfrak{N}$., then $f^{-1}(K)$ is a subset of M . By applying E2 , we get $f\left(\mathfrak{N}\beta_{PC}cl\left(f^{-1}(K)\right)\right) \subseteq \mathfrak{N}\beta cl\left(f\left(f^{-1}(K)\right)\right) \subseteq \mathfrak{N}\beta cl(K)$. it follows that $\mathfrak{N}\beta_{PC}cl\left(f^{-1}(K)\right) \subseteq f^{-1}\left(\mathfrak{N}\beta cl(K)\right)$.

E3 ⇒ E1 . Let F be a Nano β– closed set in M, then by applying E3 , $\mathfrak{N}\beta_{PC}cl\left(f^{-1}(F)\right) \subseteq f^{-1}\left(\mathfrak{N}\beta cl(F)\right)$. Since F is Nano β – closed, then $\mathfrak{N}\beta_{PC}cl\left(f^{-1}(F)\right) \subseteq f^{-1}(F)$ consequently, $f^{-1}(F)$ is Nano β_{PC} – closed set in . Hence f is $\mathfrak{N}\beta_{PC}$ – continuous function.

Proposition 3.5 For a function $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^*_{RL_{eq}}(Y)\right)$ the following statements are equivalent:

B1) f is Nanoβ_{PC} – continuous

B2) $f^{-1}\left(\mathfrak{N}\beta int(J)\right) \subseteq \mathfrak{N}\beta_{PC} - int\left(f^{-1}(J)\right)$, for each subset J of N .

B3) $\mathfrak{N}\beta int\left(f(K)\right) \subseteq f\left(\mathfrak{N}\beta_{PC} - int(K)\right)$ for any subset K in $\mathcal{M}$

Proof: $B1 \Longrightarrow B2$ Let J be any subset of $\mathfrak{N}$ .Top. S$\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$, then $\mathfrak{N}/_J$ is Nano subset of , since f is Nano P_{Pc} – continuous function ,then by proposition $E3$ we get $\mathfrak{N}\beta_{PC}cl\left(f^{-1}\left(\mathfrak{N}/_J\right)\right) \subseteq f^{-1}\left(\mathfrak{N}\beta cl\left(\mathfrak{N}/_J\right)\right) \Leftrightarrow \mathfrak{N}\beta_{PC}cl\left(\mathcal{M}/_{f^{-1}(J)}\right) \subseteq$ $f^{-1}\left(\mathfrak{N}/_{\mathfrak{N}\beta int(J)}\right) \Longrightarrow \mathcal{M}/\mathfrak{N}\beta_{PC} - int\left(f^{-1}(J)\right) \subseteq \mathcal{M}/_{f^{-1}\left(\mathfrak{N}\beta int(J)\right)}$ $\Longrightarrow$ $f^{-1}\left(\mathfrak{N}\beta int(J)\right) \subseteq \mathfrak{N}\beta_{PC} - int\left(f^{-1}(J)\right)$.

$B2 \Longrightarrow B3$. Let K be any subset of $\mathfrak{N}$.. Top. S$\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$, then $f(K)$ is a subset of N . Top .S $\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$. By applying $B2$ we get $\Longrightarrow f^{-1}\left(\mathfrak{N}\beta int\left(f(K)\right)\right) \subseteq \mathfrak{N}\beta_{PC} - int\left(f^{-1}\left(f(K)\right)\right) \Longrightarrow f^{-1}\left(\mathfrak{N}\beta int\left(f(K)\right)\right) \subseteq \mathfrak{N}\beta_{PC} - int(K)2$ $\mathfrak{N}\beta int\left(f(K)\right) \subseteq f\left(\mathfrak{N}\beta_{PC} - int(K)\right)$.

$B3 \Longrightarrow B1$. Let T be a Nano β – open subset of $\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$,then $f^{-1}(T)$ is a subset of $\left(\mathcal{M}, \tau_{RL_{eq}}(X)\right)$. By applying $B3$ we get $\mathfrak{N}\beta int\left(f\left(f^{-1}(T)\right)\right) \subseteq f\left(\mathfrak{N}\beta_{PC} - int\left(f^{-1}(T)\right)\right) \Longrightarrow \mathfrak{N}\mathcal{P}int(T) \subseteq f\left(\mathfrak{N}\beta_{PC} - int\left(f^{-1}(T)\right)\right)$ but $\in \mathfrak{N}\beta o\left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$, therefor $T \subseteq f\left(\mathfrak{N}\beta_{PC} - int\left(f^{-1}(T)\right)\right)$. Consequently, $f^{-1}(T) \subseteq \mathfrak{N}\beta_{PC} - int\left(f^{-1}(T)\right)$ .So $f^{-1}(T)$ is Nano $\mathfrak{N}\beta_{PC}$ – open. Hence f is $\mathfrak{N}\beta_{PC}$ – continuous

Definition 3.6 A function $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathfrak{N}, \tau^{*}_{RL_{eq}}(Y)\right)$ is named pre β – regular function if $f^{-1}(E)$ is Nano pre – regular in M , for every Nano β – open set E in $\mathfrak{N}$.

Remark 3.7 Every Nano Pre – regular function is $\mathfrak{N}\beta_{PC}$ – continuous.

Proposition 3.8 Every $\mathfrak{N}\beta_{PC}$-continuous function is $\mathfrak{N}\beta$-Irresolute

Proof: follows from Definition 3.1, and definition 2.17.

The converse of above statement may not be true as showing in the following

Example 3.9 Let $\mathcal{M} = \{x, y, z, w\}, \mathcal{M}/_R = \{\{x\}, \{y, z\}, \{w\}\}$ $and\ X = \{x, z\}, as\ well\ as\ \tau_R(X) = \{M, \emptyset, \{x\}, \{y, z\}, \{x, y, z\}\}$. Let $f\colon \left(\mathcal{M}, \tau_{RL_{eq}}(X)\right) \longrightarrow$

$\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$ be an identity function, then f is $\mathfrak{N}\beta$-Irresolute but it is not $\mathfrak{N}\beta_{PC}$-continuous function.

Also, not every $\mathfrak{N}\beta$-continuous function is $\mathfrak{N}\beta_{PC}$-continuous

Example 3.10 Let $\mathcal{M} = \{a,b,c,d\}, \mathcal{M}/_R = \{\{a\},\{b\},\{c\},\{d\}\}$ *and* $X = \{c,d\}$, *as well as* $\tau_R(X) = \{M,\emptyset,\{c,d\}\}$. Assume that $\mathcal{H} = \{a,b,c,d\}$ equipped with $,\mathcal{H}/_R = \{\{a,c\},\{b\},\{d\}\}$ and $Y = \{b,d\}$, then $\mathfrak{I}_{RL_{eq}}(Y) = \{\phi,\mathcal{H},\{b,d\}\}$ .Define $g\colon\left(\mathcal{M},\tau_{RL_{eq}}(X)\right) \longrightarrow \left(\mathcal{H},\tau_{RL^*{}_{eq}}(Y)\right)$ by $g(a) = a$, $g(b) = b$, $g(c) = c$, $g(d) = d$, then g is $\mathfrak{N}\beta$ -continuous function .but $g^{-1}(\{b\}) = \{b\} \notin \mathfrak{N}\beta_{PC}O\left(\mathcal{M},\tau_{RL_{eq}}(X)\right)$, for $\{d\} \in \mathfrak{N}\beta o\left(\mathcal{H},\tau_{RL^*{}_{eq}}(Y)\right)$. Hence it is not $\mathfrak{N}\beta_{PC}$-continuous.

Conclusions

The main purpose of this work is to define a new type of Nanoβ-open namely Nano-β_{PC} open set, which is a strong form of Nano β-open. Additionally, we identify a class of Nano β_{PC}-open sets in terms of upper approximation, lower approximation and boundary region. Furthermore, it has a number of given properties. In addition, we introduce a completely novel kind of Nano continuity that we've labeled Nano β_{PC}-continuous functions, and give several characterizations about this function.

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