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Exploring Solutions of Geometry Problems for Inverse Cauchy Problems in Helmholtz and Modified Helmholtz Equations

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Abstract

In the present paper explores a reverse Cauchy problem for a heat transfer issue described by the Helmholtz and modified Helmholtz equation. Our goal is to identify an unknown defect D within a simply connected bounded domainΩ, given the Dirichlet data (temperature) h on the boundary∂D, and Neumann data (heat flux) $\partial_n h$ on the boundary$\partial\Omega$. We assume that the temperature h satisfies the Helmholtz equation (or modified Helmholtz equation) that governs the heat condition in the fin. To solve this problem, we propose a method that involves two steps. First, we solve a Cauchy problem using the Helmholtz equation (or modified Helmholtz equation) to determine the temperature$h$. Then, in the second phase, we solve a system of nonlinear scalar equations to determine the coordinates of the points defining the boundary∂D. This can be achieved using an iterative method, such as Newton's method.

Keywords: Inverse Cauchy problem, Helmholtz equation, modified Helmholtz equation, polynomial expansion, Newton method.

Introduction

In This article involves on the inverse problem of identifying an anonymous defect D within a bounded domainΩ, characterized by boundary $\Gamma = \partial\Omega$ (boundary temperature and heat flux). It assumes that the steady state temperature h gratifies the Helmholtz equation (or the modified Helmholtz equation), which governs heat conduction in a fin:

$$\Delta h \pm k^2 h = F, \qquad \Omega/D$$

with the domain being $\Omega\backslash D \subset \mathbb{R}^2$ and F being a function that is provided. Any problem of this nature is an example of an ill-posed problem. In point of fact, a problem is considered to be well-posed in the sense of Hadamard if the solution is one that is both unique and stable [Hadamard, 1923]. On the other hand, if the solution does not satisfy any of these three conditions, then the problem is considered to be ill-posed, and in order to solve this ill-posed problem, an inverse problem needs to be formulated. When compared to the direct difficulties, the inverse problem is often considered to be more challenging to solve than the direct problems.

Additionally, the inverse problems exhibit instability, as noted by [Hadamard in 1923]. This means that even a slight error in the input data measurement can result in a significant perturbation or error in the solution.

In this article, we adopt a technique akin to the approach employed in citations [1, 3, 4, 14, 11, 19, 24] to estimate the solution through a polynomial expansion. Moreover, we make use of the discrete decoupled Cauchy-Newton algorithm [20] to precisely ascertain the unidentified interior boundary. The creators of the mentioned study employed this methodology to tackle an inverse problem related to the modified Helmholtz equation. However, in our study, we applied it to solve an inverse problem for the Helmholtz equation and compare the results with those obtained for the modified Helmholtz equation, considering different physical parameters$k = \sqrt{2}$,$\sqrt{15}$,$\sqrt{25.5}$,$\sqrt{52}$.

In the remainder of the parts of this paper, the following is included: In the second section, we take the Helmholtz equations and the modified Helmholtz equations together. In reference to the Decoupled Cauchy-Newton method, which was discussed in section three. In the fourth section, you will demonstrate the numerical results and the discussion by providing some examples. At long last, we discuss our findings and conclusions.

Mathematical Expression for Helmholtz equation and modified Helmholtz equation

Consider the assumption that there exists a bounded domain Ω in $\mathbb{R}^2$ that is simply linked and has a smooth border$\partial\Omega$. Let D be an unidentified defect that is compactly contained inside Ω and has a border that is∂D. It is assumed that this defect is smooth and piecewise analytic, and that it is linked at the point where $\Omega_a = \Omega \backslash D$. Under the assumption that the temperature $h(x, y)$ in Ω_a is in accordance with the modified Helmholtz equation and the Helmholtz equation:

$$\Delta h(x, y) \pm k^2 h(x, y) = F\ (x, y), \qquad (x, y) \in \Omega_a \tag{1}$$

That is, the value of k is determined by the convective heat transfer coefficient, the thermal conductivity, and the thickness of the fins. It is assumed that the steady state Dirichlet boundary conditions on the interior and outer borders ∂D and $\partial\Omega$ are supplied, which means that the Dirichlet temperature data $h\backslash_{\partial\Omega a}$ is given, and the temperature h is able to satisfy the following problem:

$$\Delta h(x, y) \pm k^2 h(x, y) = F(x, y)\ , \qquad (x, y) \in \Omega_a \tag{2}$$

$$h(\delta, \theta) = f(\delta, \theta)\ , \qquad (\delta, \theta) \in \partial\Omega \tag{3}$$

$$h(\delta, \theta) = h_0(\delta, \theta), \qquad (\delta, \theta) \in \partial D \tag{4}$$

where F, f and h_0 are given function

When D is known, it is common knowledge that the direct Dirichlet problem, which is represented by equations (2)–(4), has a singular solution, $h \in H^1(\Omega\backslash D)$ .and it is known that $F \in L^2(\Omega)$, $f \in H^{\frac{1}{2}}(\partial\Omega)$, and $h_0 \in H^{\frac{1}{2}}(\partial D)$ [see 21] .In addition to this, we are subject to the following condition:

$$\partial v\ h(\delta, \theta) = g\ (\delta, \theta), \tag{5}$$

where $\partial\Omega$ represents the portion of the boundary that is accessible, and $\partial_v h\backslash_{\partial\Omega}$ represents the Neumann heat flow data on the boundary defined by $\partial\Omega$.Therefore, the inverse issue that is being considered involves the extraction of certain information on the boundary ∂D from the data $g|_{\partial\Omega}$. and instead of measuring the flux on the full boundary $\partial\Omega$, the data (5) might only be partial, meaning it only measures a non-zero section $\Gamma \subset \partial\Omega$.

It is common knowledge that the inverse problem is described as being nonlinear and ill-posed, in contrast to the direct problem, which is described as being linear and well-posed. While it has been demonstrated, as referenced in [13] that the solution to the inverse problem outlined in Equations. (2)– (4) and (5) is unique, this solution is not continuously dependent on errors in the input Cauchy data (3) and (5).

Decoupled Cauchy-Newton algorithm

The using Nachaoui-Aboud decoupled Cauchy-Newton method [20], we will solve the equation (2-5) in order to locate D. In order to solve the inverse issue and find the temperature h in D, we must first solve a Cauchy problem that is controlled by the Helmholtz equation and the modified Helmholtz equation. Polynomial expansion is the method that we use when dealing with situations that have smooth boundaries, By using this approach, we will get a solution for h that is based on polynomial expansion. Once we have a solution for h, we go on to the second sub-problems, which include determining the coordinates of the points that define the boundary D. This is done after we have determined the answer for h. The application of a number of nonlinear scalar equations is what we do in order to get these results. Before we can obtain the nonlinear equations, we must first convert the problem into a parametric form and then solve it numerically by using an iterative technique like the Newton method.

Polynomial Expansion approximation of the Cauchy problem

In order to get a close approximation of the solution to the Cauchy problem (2)– (5), we offer a polynomial expansion that is comparable to the one that was presented in [1, 3, 11, 14, 24]. We specifically need an approximation polynomial that follows the general form:

$$h(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{i} c_{ij}\, x^{i-j}\, y^{j-1} \tag{6}$$

When the coefficients c_{ij} are unknown and to be found, with $1 \leq i \leq m$ and $1 \leq j \leq i$, the total number of these unknown coefficients is $\frac{m(m+1)}{2}$. With this formula substituted into the Helmholtz equations and the Helmholtz equations changed in (2), we would get:

$$\sum_{i=1}^{m}\sum_{j=1}^{i}(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \pm k^2)c_{ij}x^{i-j}y^{j-1} = F(x,y\,) \tag{7}$$

We can write the previous equation as follows:

$$\sum_{i=1}^{m}\sum_{j=1}^{i}(\frac{(i-j)(i-j-1)}{x^2}+\frac{(j-1)(j-2)}{y^2} \pm k^2)\,c_{ij}x^{i-j}y^{j-1} = F(x,y) \tag{8}$$

For each of the n_{1b} points, we obtain a system of linear equations with coefficients c_{ij} by noting that this equation must be satisfied for every (x,y) in the domain Ω.

$$\sum_{i=1}^{m}\sum_{j=1}^{i}(\frac{(i-j)(i-j-1)}{x_l^2}+\frac{(j-1)(j-2)}{y_l^2} \pm k^2)c_{ij}x_l^{i-j}y_l^{j-1} = F(x_l,y_l) \tag{9}$$

for $l = 1,\ldots,n_{1b}$.

We must define boundary conditions for h to finalize this system. Given that we have Dirichlet and Neumann boundary information, we can utilize them to compute the coefficients c_{ij}. It is important to mention that the normal derivative of $u(x,y)$ can be expressed as shown in [17],

$$\partial_n u = \eta(\theta)\left[\cos(\theta) - \frac{\rho'}{\rho^2}\sin(\theta)\right]\partial_x u + \eta(\theta)\left[\sin(\theta) - \frac{\rho'}{\rho^2}\cos(\theta)\right]\partial_y u \tag{10}$$

in which $\eta(\theta)$ have been defined as:

$$\eta(\theta) = \frac{\rho(\theta)}{\sqrt{\rho^2(\theta) + [\rho'(\theta)]^2}} \tag{11}$$

Now, by defining h as u and ρ as δ, and utilizing formula (10) for the normal derivative in boundary condition (5), and replacing the solution h in (3) with its approximation from (6), we obtain a system from equations. Evaluating this system at n_{1a} locations yields a system of

$2 * n_{1a}$ linear equations .if this system is combined with the n_{1b} equations from (9), a system of linear equations of the form is obtained.

$$Ac = b \tag{12}$$

where the vector c is of length

$$n_2 = \frac{m(m+1)}{2} \tag{13}$$

The coefficients c_{ij} that need to be calculated have been rearranged within it. The rectangular matrix A in the system has dimensions $n_1 * n_2$ and a length given$n_1 = 2 * n_{1a} + n_{1b}$. The known data vector b also has a length of $n_1 = 2 * n_{1a} + n_{1b}$.

This means that solving the system of linear algebraic equations (12) is all that's needed to resolve the inverse Cauchy problems (2), (3), and (5). To discover the unique solution of the system of linear algebraic equations (12), the number of collocation points n_1 and the number n_2 have to satisfy the inequality$n_1 \geq n_2$. After the coefficients c_{ij} have been discovered, the polynomial expansion (6) can be utilized to evaluate the solution $h(x,y)$ of the Cauchy problem (2)-(5) at points within the domainΩ.

A Represents the matrix that defines the linear system resulting from the polynomial expansion. ***b*** is the represents data vector, ***c*** represents the vector of coefficients in the polynomial expansion, and the regularization parameter is a small positive constant.

Determining the unknown Boundary with Newton's Method

Through the utilization of the Nachaoui-Aboud discrete decoupled Cauchy-Newton algorithm [20], it is assumed that the Cauchy problem has been resolved. This is because the vector c allows for the evaluation of the function $h(x,y)$ at any point (x,y) in Ω. We can assert that for a fixedθ, there exists a value $\delta(\theta)$ in the interval $(0, \delta_0(\theta))$ such that $\delta(\theta)$ is on the boundary$\partial\Omega$, and we can continue this process to uncover the unknown boundary ∂D. With the boundary condition (4) on$D$, we can define a point $(\delta(\theta), \theta)$ on ∂D for a given θ in the range$[0,2\pi]$.

$$h(\delta, \theta) = h_0(\delta, \theta) \tag{14}$$

where h is the answer to the Cauchy problem that is shown by (2)–(3) and (5). We describe

$$F_{rN}(\delta(\theta)) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij} \left(\delta(\theta)\right)^{i-1} (\cos(\theta))^{i-j} (\sin(\theta))^{j-1} - h_0(\delta(\theta), \theta) = 0 \tag{15}$$

It is essential to calculate the root $\delta(\theta)$ of the nonlinear operator F_{rN} as defined by the nonlinear equation (15) in order to ascertain the approximate value of $(\delta(\theta), \theta) \epsilon \partial D$, where the boundary condition (4) is satisfied. The reason for this is because equation (15) is a scalar nonlinear equation. The solution for the root of F_{rN}, given a fixed $\theta \epsilon [0,2\pi]$, can be determined using Newton's method. Since the F_{rN} operator is explicitly defined in terms of $\delta(\theta)$.Note that:

$$d_\delta Fr_N\big(\rho(\theta)\big) = \sum_{i=1}^{m} \sum_{j=1}^{i} c_{ij}(i-1)\big(\delta(\theta)\big)^{i-2}(\cos(\theta))^{i-j}(\sin(\theta))^{j-1} - d_\delta h_0(\delta(\theta),\theta) \tag{16}$$

This means that $d_\delta F_{rN}$ is the derivative of the function F_{rN} with respect to δ.

With Newton's algorithm, we generate a series of values denoted by δ_n, starting with the initial value $\delta_0 = \delta_0(\theta)$, and then updating it as follows:

$$\delta_{n+1} = \delta_n - \frac{Fr_N(\delta_n)}{d_\delta Fr_N(\delta_n)} \tag{17}$$

Consequently, we can summarize our method for solving the inverse geometric problem for the Helmholtz and modified Helmholtz equations (2)-(5), This involves determining the inner boundary of an annular domain based on a given set of boundary Cauchy data (temperature and heat flux) using the Nachaoui-Aboud discrete decoupled Cauchy-Newton algorithm [20].

Numerical Results and Discussion

This section presents the numerical results obtained using the polynomial approximation method described and the Nachaoui-Aboud discrete decoupled Cauchy-Newton algorithm. These findings are utilized to address the Cauchy problem for the Helmholtz and modified Helmholtz equations in two-dimensional bounded domains. By employing the Conjugate Gradient Least Squares (CGLS) method, the data that was obtained from the linear system that was represented in equation (12) is resolved.

Example 1:

Assume that we have the Cauchy problem for (Helmholtz & modified Helmholtz equations) with exact solution:

$$h(x,y) = 6x^2y^2 - x^4 - y^4$$

The Dirichlet data given on the interior and exterior boundaries ∂D and $\partial\Omega$ as follows:

$$h(\delta,\theta) = 6r^4(\cos(\theta)\sin(\theta))^2 - (r\cos(\theta))^4 - \big(r\sin(\theta)\big)^4$$

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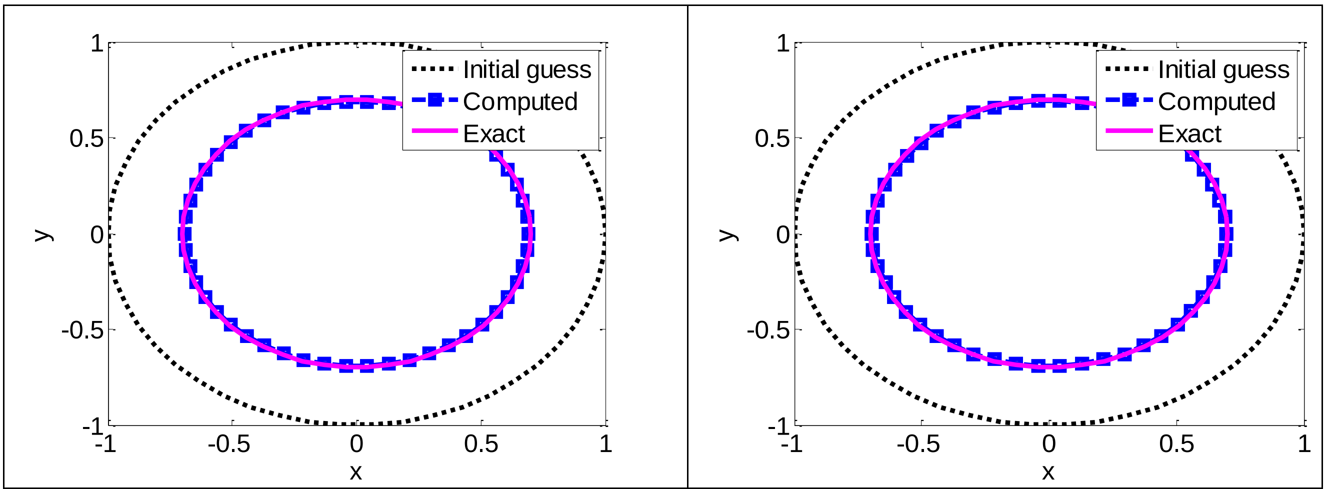
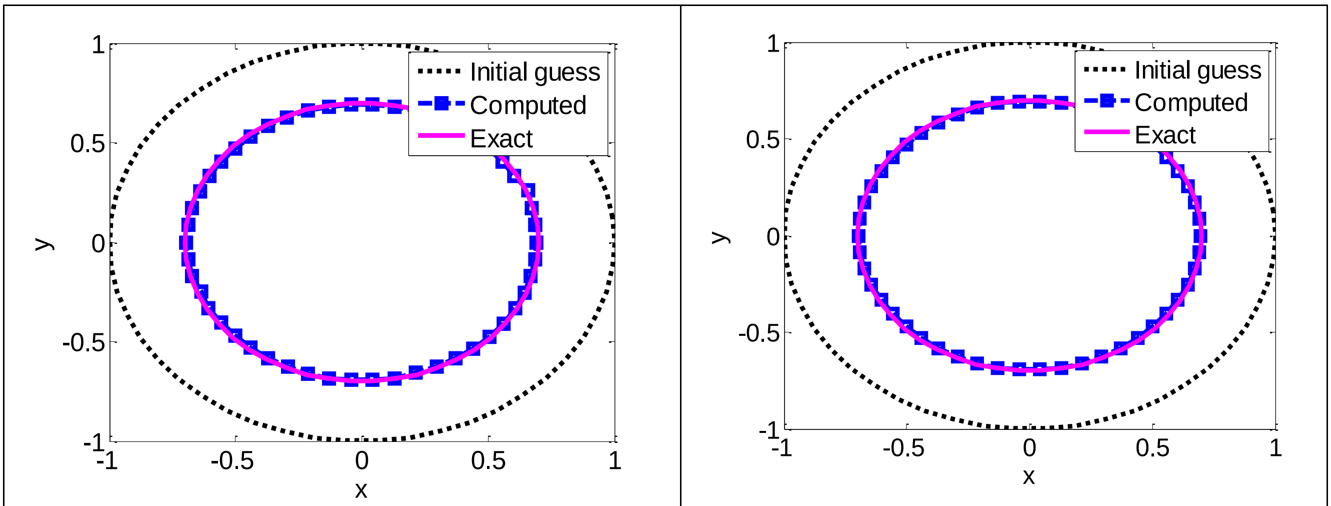
$$g(\theta) = \frac{\partial h}{\partial n} = (12xy^2 - 4x^3)\cos(\theta) + (12yx^2 - 4y^3)\sin(\theta)$$

With domain Ω say circle with radius (0.7), For decouple Cauchy-Newton, we take $n_{1a} = 50$,and for stopping criterion in the Newton iteration $|\delta_{n+1} - \delta_n| \le 10^{-2}$ we take the cases that $k = \sqrt{2}$ and $\sqrt{15}$,where k is a physical parameter. The results represent as follows:



$\boldsymbol{k = \sqrt{2}}$	$\boldsymbol{= \sqrt{15}}$

a:results for Helmholtz equation



$\boldsymbol{k = \sqrt{2}}$	$\boldsymbol{k = \sqrt{15}}$

b:results for modified Helmholtz equation

Figure 1: From figures it is clear that the approximation results with a high degree of precision.

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Example 2: Suppose that the boundary conditions in (2), (3) are computed from the exact solution:

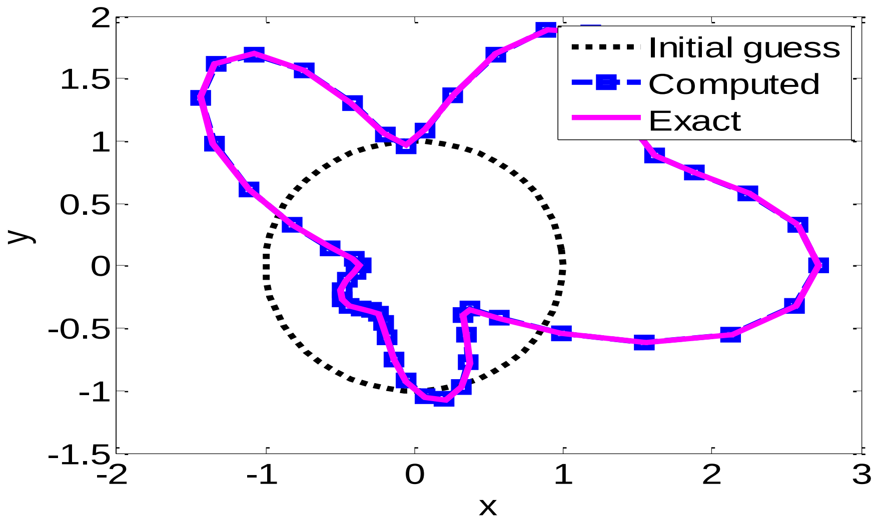
$$h(x,y) = x^4 - y^4$$

This equation satisfied the Helmholtz (or modified Helmholtz) equation and its given as follows:

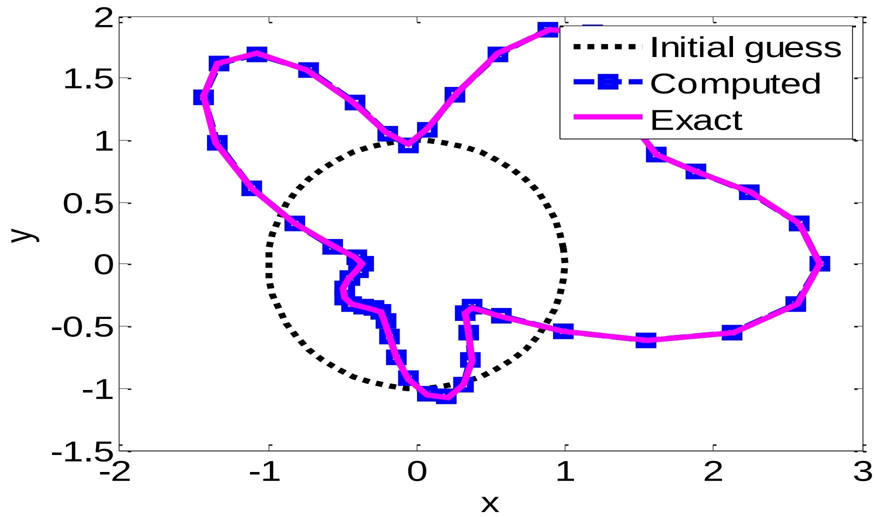
$$h(\delta,\theta) = (r(\cos(\theta))^4 - \big(r(sin\theta)\big)^4$$

$$g(\theta) = \frac{\partial h}{\partial n} = 4x^3(\cos(\theta)) - 4y^3(\sin(\theta))$$

For decoupled Cauchy-Newton algorithm, we take $= \sqrt{25.5}$, $\sqrt{52}$, $n_{1a} = 50$, and $Tol = 10^{-4}$, the results represent as follow :



a. $\boldsymbol{k} = \sqrt{\mathbf{25.5}}$



b. $\boldsymbol{k} = \sqrt{\mathbf{52}}$

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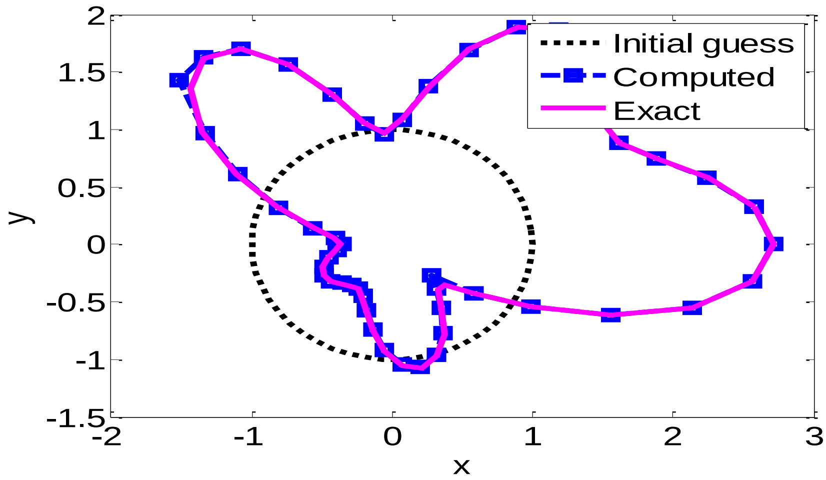
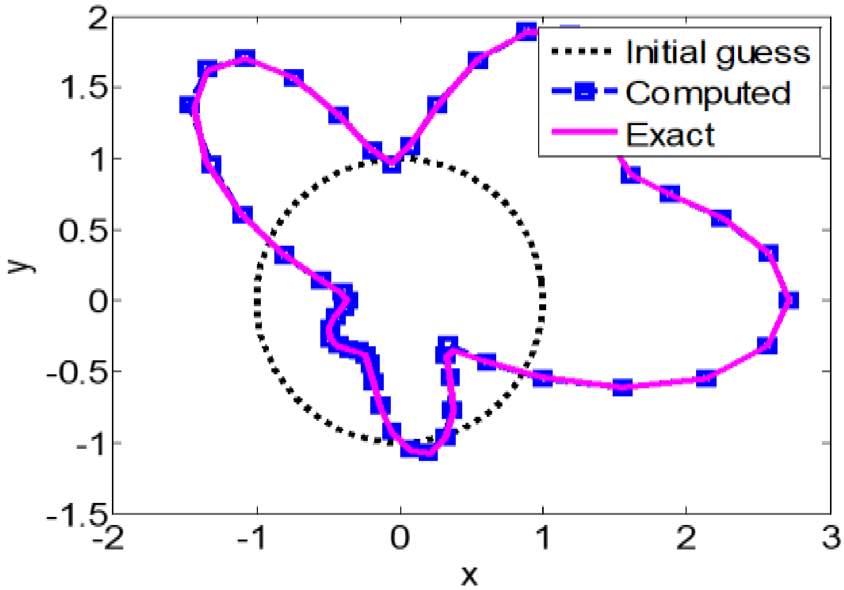


c. $k = \sqrt{25.5}$



d. $k = \sqrt{52}$

Figure 2: The results in a. and b. with Helmholtz equation and c. and d. with modified Helmholtz equation.

These outcomes are acquired through initial iteration of Newton's technique carried out on the circle.

Noise Effect

In practical applications, when inverse problems are formulated, known data is acquired through using measurements that may leading to mistakes. To assess the stability of the numerical solution, we introduce noise into the Dirichlet and Neumann boundary data f and g provided by equations (4) and (5).

The following equation is used to perturb the precise Cauchy data:

$$f(\theta) = h_{ex}(\delta(\theta), \theta) + Noise_{Dirichlet}$$

$$g(\theta) = \partial h_{ex}(\delta(\theta), \theta) + Noise_{Neumann}$$

In this case, $Noise_{Dirichlet}$, $Noise_{Neumann}$ represents the randomly distributed Gaussian error, and it refer to the noise level with a values of 0.001, 0.01, 0.05, and 0.1

Example 3: We take into account all provided data in (2), (3) which were taken similarly in example 2. The boundary which was acquired using 0.01, 0.001, 0.05, 0.1 noise consecutively, for both Dirichlet and Neumann boundary data utilizing parameter physical $k = \sqrt{52}$ the outcomes was illustrated as shown :

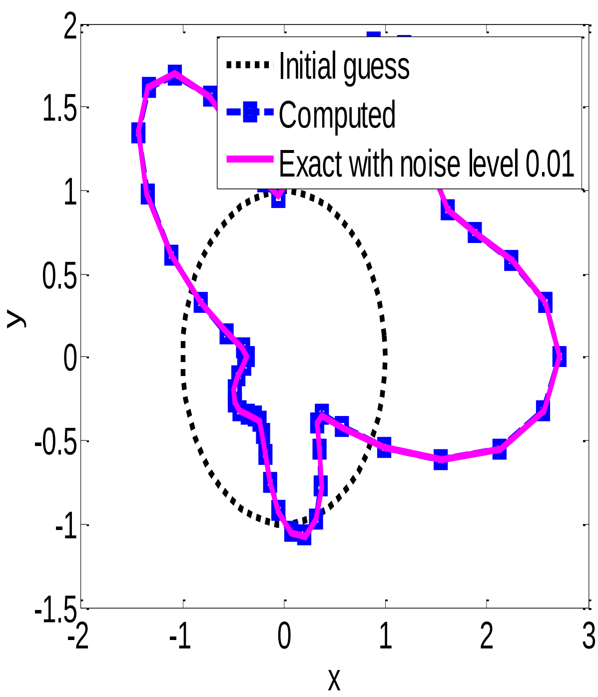


Figure 3(a): With $Noise_{Dirichlet} = 0.01$

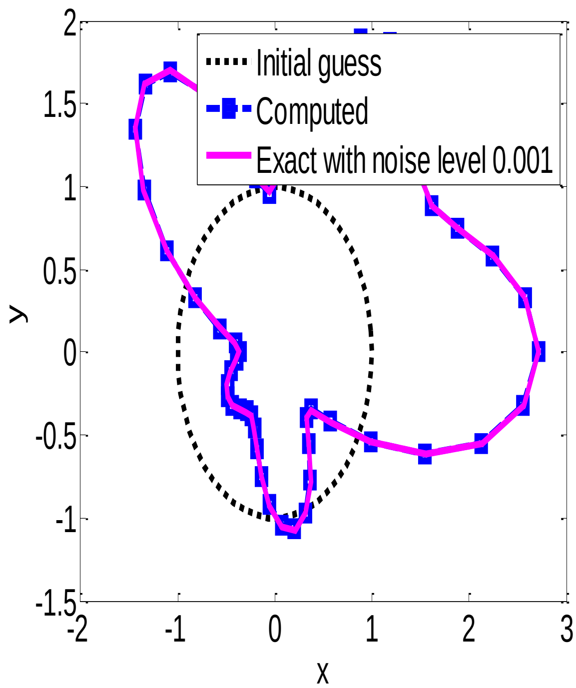


Figure 3(b): With $Noise_{Dirichlet} = 0.001$

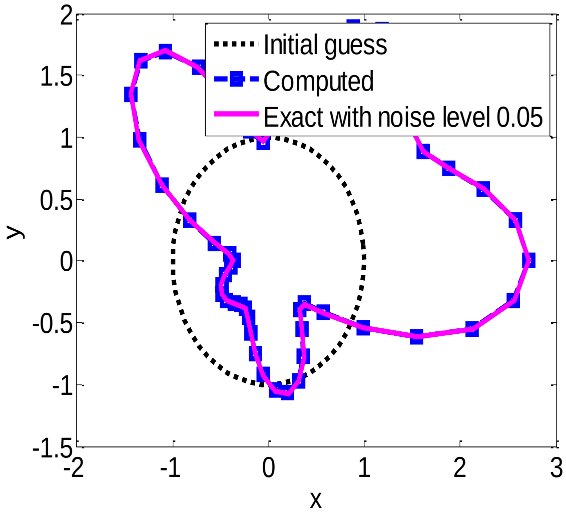


Figure 3(c): With $Noise_{Dirichlet} = 0.05$

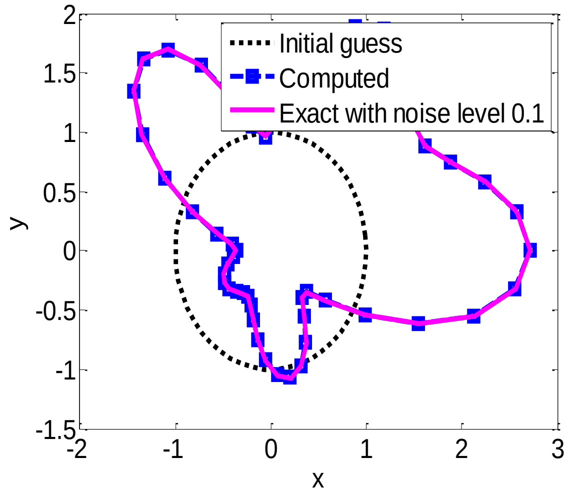


Figure 3 (d): With $Noise_{Dirichlet} = 0.1$

Figure 3: The effect of noise with Helmholtz equation. (Effect of noise on Dirichlet boundaries)

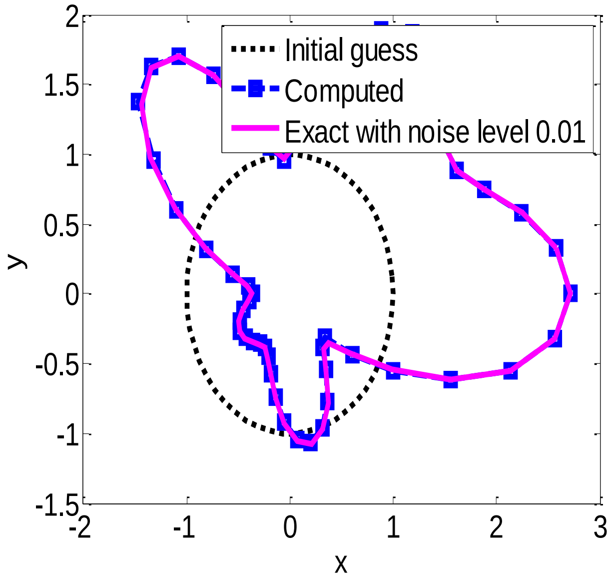


Figure 4(a): With $Noise_{Dirichlet} = 0.01$

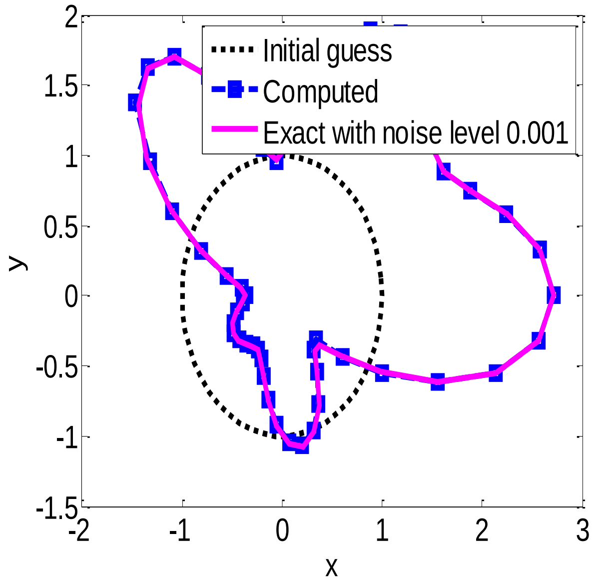


Figure 4(b): With $Noise_{Dirichlet} = 0.001$

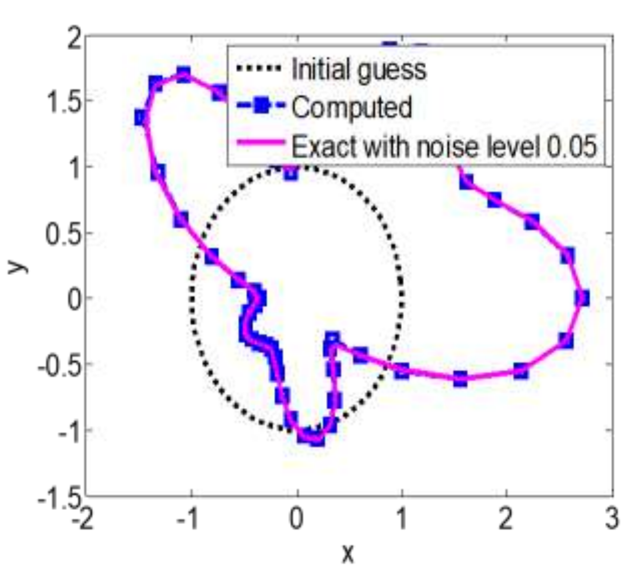


Figure 4(c): With $Noise_{Dirichlet} = 0.05$

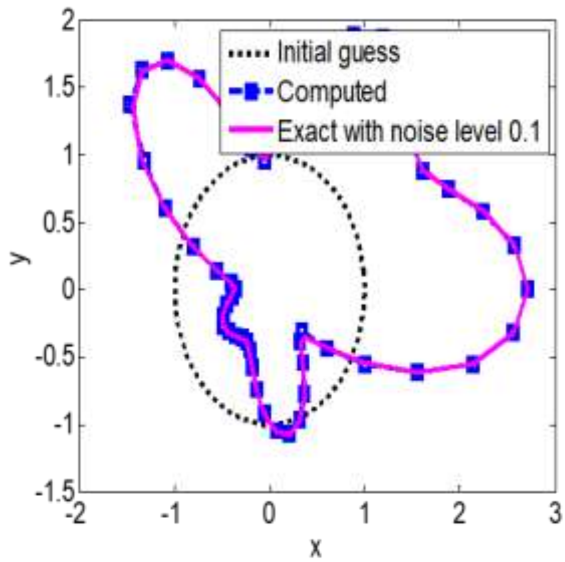


Figure 4(d): With $Noise_{Dirichlet} = 0.1$

Figure 4: The effect of noise with modified Helmholtz equation. (Effect of noise on Dirichlet boundaries)

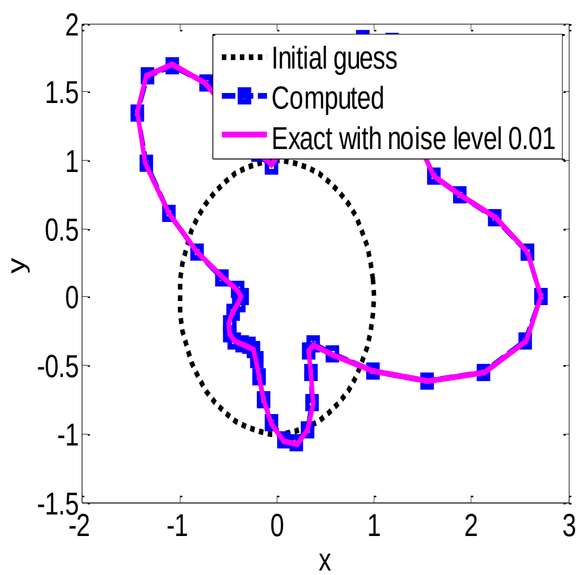


Figure 5(a):With $Noise_{Neumann} = 0.01$

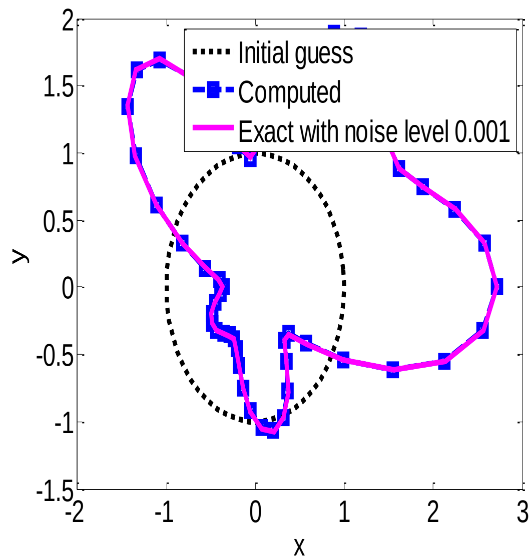


Figure 5(b):With $Noise_{Neumann} = 0.001$

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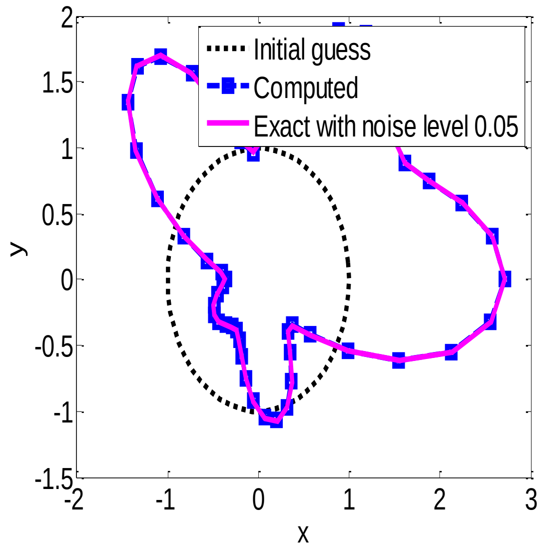

Figure 5(c):With $Noise_{Neumann} = 0.05$

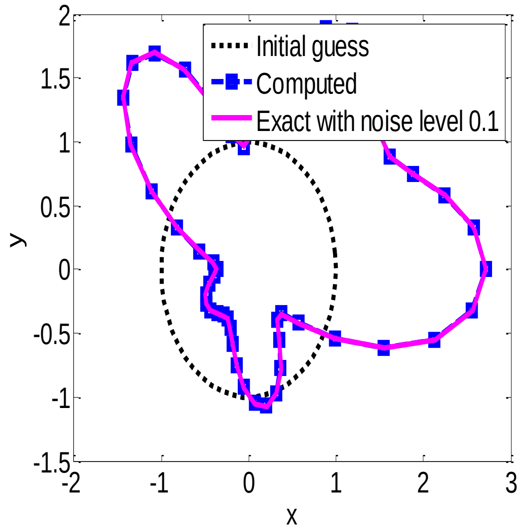

Figure 5(d):With $Noise_{Neumann} = 0.1$

Figure 5: The effect of noise with Helmholtz equation. (Effect of noise on Neumann boundaries)

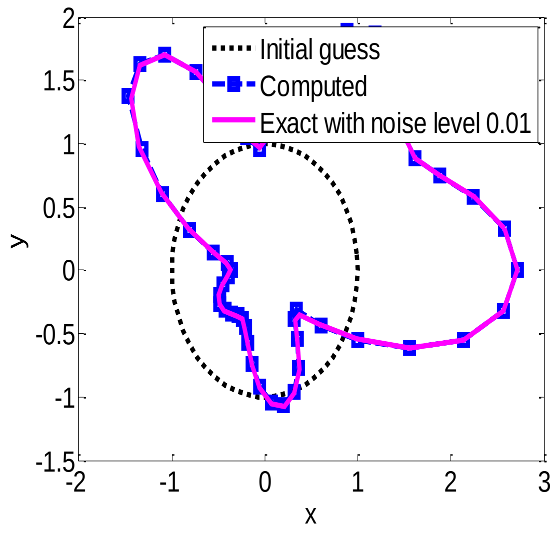

Fig6(a):With $Noise_{Neumann} = 0.01$

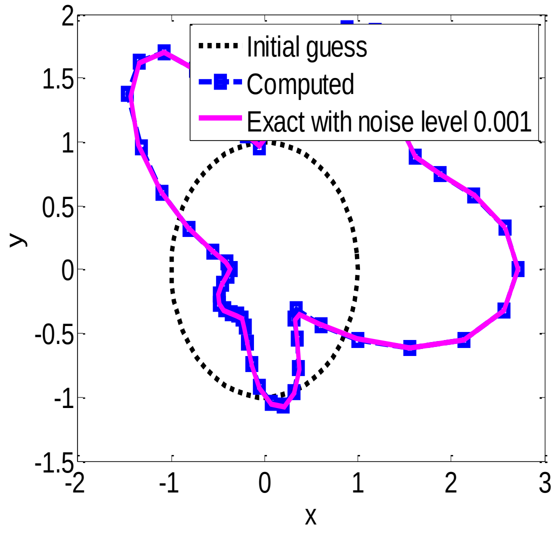

Fig6(b):With $Noise_{Neumann} = 0.001$

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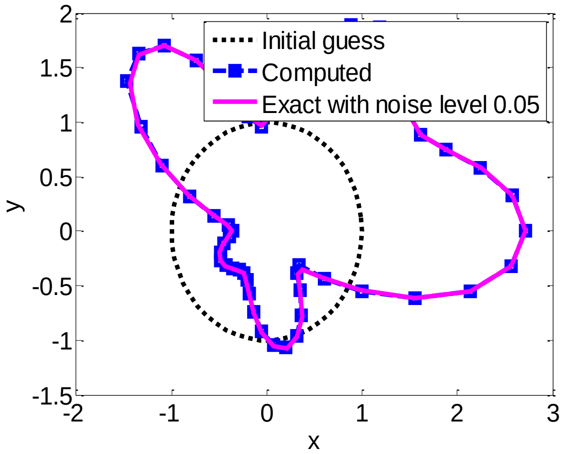

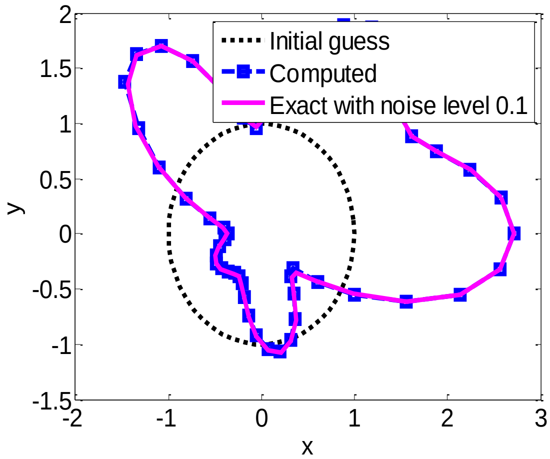

Figure 6(d): With $Noise_{Neumann} = 0.1$ **Figure 6(c):With** $Noise_{Neumann} = 0.05$

Figure 6: The effect of noise with modified Helmholtz equation. (Effect of noise on Neumann boundaries)

Conclusion

This study aims to address an inverse Cauchy issue related to a heat transfer problem that is ruled by the Helmholtz equation and a modified Helmholtz equation. The goal is to ascertain the unidentified defect D within a bounded domain Ω that is simply connected. This is accomplished by utilizing the Dirichlet data (temperature) h at the border ∂D and the Neumman data (heat flux) $\partial_n h$ at the border $\partial\Omega$. The method that is being proposed involves breaking the problem down into two sub-problems. The first stepping a Cauchy problem is solved by using the Helmholtz equation (or a modified Helmholtz equation) to determine the temperature h. Then, it uses a polynomial expansion to get a close approximation of the solution, which lets us get the immediate problem in solving a linear system, which is addressed by the CGLS algorithm. The other stepping in resolving a nonlinear scalar equation involves determining the positions of the points that establish the boundary by utilizing an iterative technique like Newton's method, and the both Helmholtz equation and its modified Helmholtz equation were investigated, with varying examples studied for each, the approximate value of the precise limit is determined with high precision for the various in boundaries (whether they are regular or irregular boundary ∂D). Certainly, when considering diverse boundary variations and various precise solution types (including polynomial as well as non-polynomial), obtaining a close approximation of the boundary's exact value becomes, achieved with a geometry resembling the exact one. Since the inverse problems are inherently unstable, an alternative approach is to introduce noise into the Cauchy data, that way verifying the stability of the solution.

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