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Generalized Derivations of BF-Algebras

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Abstract

In this paper, we introduced the concept of generalized derivations of an algebraic structure namely BF-algebras and studied some of its basic properties. Moreover, some results that concern on the kernel of the generalized derivations of a BF-algebras are presented. Furthermore, we studied the concept of torsion free on a BF-algebras.

Keywords: BF-algebras, Derivations, Generalized derivations, B-algebras, BCI-algebras.

تعميم الاشتقاقات على جبر-BF

محمد خالد شاحوذ

وزارة التربية - المديرية العامة للتربية في الرمادي

الخلاصة

في هذا البحث قدمنا فكرة تعميم الاشتقاقات لتركيبة جبرية تسمى جبر-BF ودرسنا بعض خصائصها الاساسية. علاوة على ذلك, تم تقديم بعض النتائج المتعلقة بنواة تعميم الاشتقاقات لجبر-BF. اضف الى ذلك, درسنا مفهوم جبر-BF طليق الالتواء. بعض الامثلة المختلفة تم عرضها لتوضيح بعض النتائج.

الكلمات المفتاحية : جبر-BF, الاشتقاقات, تعميم الاشتقاقات, جبر-B, جبر-BCI.

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Introduction

The concept of derivations is one of the influential subjects in rings and near rings. This concept has been studied in various types of rings, such as $*$-ring and Γ-ring in order to study some of their properties. The study of this concept has been extended to some other algebraic structures by using an appropriate approach that can make its study is possible. Some new algebraic structures have been presented by using the concept of derivations and studying some of their properties. For instance, Al-Shehrie [1] applied the concept of derivations on B-algebras and studied some of its properties such as a properties of (l, r)-derivations (resp. (r, l)-derivations) of B-algebras and the properties of derivations of zero-commutative B-algebras. Moreover, the concept of derivations of UP-algebras has been defined by Sawika et. al [2]. Then, some of its basic properties have been investigated. In addition, the same concept has been defined on BF-algebras by Dejen and Tegegne [3], then some of its properties have been discussed.

On the other hand, some other studies have been investigated the concept of generalized derivations for some algebraic structures and discussed some of its properties. For example, the generalized derivations of BCI-algebras was presented by Ozturk et. al [4], and some of its properties are discussed. A generalized left derivation of BCI-algebras was introduced in [5]. The authors showed some properties of this notion. In the same year, Kim [6] presented the generalized derivations of BE-algebras and studied some of its properties. Furthermore, the concept of generalized derivations of BCC-algebras has been introduced by Bawazeer et. al [7], and some of its properties have been investigated. In the same year, Ganeshkumar and Chandramouleeswaran [8] presented the generalized derivation of TM-algebras with some of its properties. Finally, the generalized derivation of d-algebras has been investigated by Al-omary et. al [9].

Motivated by the work of previous researchers, we presented the notion of generalized derivations of BF-algebras and studied some of its properties in this paper. The present paper has been structured as follows. In section two, some basic concepts which are important in

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this study are presented. Section three contains the main results of this paper. Section four, followed by the conclusions and future research scope of the present paper.

1. Basic Concepts

Definition 1.1[10] A non-empty set $\mathfrak{W}$ with binary operation $\circ$ and a constant 0 is said to be a BF-algebras, if the conditions below are holds.

i. $\varsigma \circ \varsigma = 0, \forall \varsigma \in \mathfrak{W}$,

ii. $\varsigma \circ 0 = \varsigma, \forall \varsigma \in \mathfrak{W}$,

iii. $0 \circ (\varsigma \circ \xi) = (\xi \circ \varsigma), \forall \varsigma, \xi \in \mathfrak{W}$.

Definition 1.2 [10] Let $\mathfrak{W}$ be a BF-algebras and let $\Phi \neq \mathfrak{H} \subseteq \mathfrak{W}$, then $\mathfrak{H}$ is called sub-algebra of $\mathfrak{W}$, if for every $\varsigma, \xi \in \mathfrak{H}$, then $\varsigma \circ \xi \in \mathfrak{H}$.

Definition 1.3 [3] Let $(\mathfrak{W}, \circ, 0)$ be a BF-algebra. A self-map $\mathcal{J}: \mathfrak{W} \longrightarrow \mathfrak{W}$ is said to be (l, r)-derivation of $\mathfrak{W}$ if for any $\varsigma, \xi \in \mathfrak{W}$, then $\mathcal{J}(\varsigma \circ \xi) = (\mathcal{J}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi))$. A self-map $\mathcal{J}: \mathfrak{W} \longrightarrow \mathfrak{W}$ is called (r, l)-derivation of $\mathfrak{W}$ if for any $\varsigma, \xi \in \mathfrak{W}$, then $\mathcal{J}(\varsigma \circ \xi) = (\varsigma \circ \mathcal{J}(\xi)) \wedge (\mathcal{J}(\varsigma) \circ \xi)$. If $\mathcal{J}$ satisfying both of (l, r)-derivation and (r, l)-derivation, then it's called a derivation of $\mathfrak{W}$.

Definition 1.4 [4] Let $\mathfrak{W}$ be a BCI-algebras. Then, $\mathfrak{W}$ is called torsion free, if for any $\varsigma \in \mathfrak{W}$ we have that $\varsigma + \varsigma = \varsigma \circ (0 \circ \varsigma) = 0 \Longrightarrow \varsigma = 0$.

2. Main Results

This section presents the concept of generalized derivations of BF-algebras with some of its basic properties. Throughout this section we mean by $\mathfrak{W}$ is a BF-algebras unless otherwise we mentioned. We commence this section by the following remark.

Remark 2.1 For a BF-algebras $\mathfrak{W}$ we meant $\varsigma \wedge \xi = \xi \circ (\xi \circ \varsigma)$ for every $\varsigma, \xi \in \mathfrak{W}$.

Definition 2.2 Let $\mathfrak{W}$ be a BF-algebras. A self- map $\mathcal{G}: \mathfrak{W} \longrightarrow \mathfrak{W}$ is said to be generalized derivation (for short G-d), if there exist associated (l, r)-derivation and (r, l)-derivation $\mathcal{J}: \mathfrak{W} \longrightarrow \mathfrak{W}$ satisfies

(i) $\mathcal{G}(\varsigma \circ \xi) = (\mathcal{G}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi))$ for any $\varsigma, \xi \in \mathfrak{W}$.

(ii) $\mathcal{G}(\varsigma \circ \xi) = (\varsigma \circ \mathcal{G}(\xi)) \wedge (\mathcal{J}(\varsigma) \circ \xi)$ for any $\varsigma, \xi \in \mathfrak{W}$.

Example 2.3 [3] Let $\mathfrak{W} = \{0, \varsigma, \xi, \zeta\}$ with the binary operation defined in the table below.

$\circ$	0	ς	ξ	ζ
0	0	ξ	ς	ζ
ς	ς	0	ζ	ξ
ξ	ξ	ζ	0	ς
ζ	ζ	ς	ξ	0

Then, $(\mathfrak{W}, \circ, 0)$ is a BF-Algebras. Define the self-map $\mathcal{J}: \mathfrak{W} \longrightarrow \mathfrak{W}$ by

$$\mathcal{J}(\mathrm{e}) = \begin{cases} \zeta, & if\ \mathrm{e} = 0 \\ 0, & if\ \mathrm{e} = \varsigma \\ \varsigma, & if\ \mathrm{e} = \xi \\ 0, & if\ \mathrm{e} = \zeta \end{cases}$$

Then, $\mathcal{J}$ is a derivation of $\mathfrak{W}$. Furthermore, define the self-map $\mathcal{G}: \mathfrak{W} \longrightarrow \mathfrak{W}$ by

$$\mathcal{G}(\mathrm{e}) = \begin{cases} 0, & if\ \mathrm{e} = \zeta \\ \varsigma, & if\ \mathrm{e} = \xi \\ \xi, & if\ \mathrm{e} = \varsigma \\ \zeta, & if\ \mathrm{e} = 0 \end{cases}$$

Then, it is easy to prove that $\mathcal{G}$ is a G-d of $\mathfrak{W}$.

Example 2.4 Consider a BF-algebras which is given in Example 2.3 with the derivation $\mathcal{J}$.
Define the self-map $\mathcal{G}: \mathfrak{W} \longrightarrow \mathfrak{W}$ by

$$\mathcal{G}(\mathrm{e}) = \begin{cases} 0, & if\ \mathrm{e} = 0 \\ \varsigma, & if\ \mathrm{e} = \varsigma \\ \xi, & if\ \mathrm{e} = \xi \\ \zeta, & if\ \mathrm{e} = \zeta \end{cases}$$

Clearly that $\mathcal{G}$ is a G-d of $\mathfrak{W}$.

Example 2.5 Consider a BF-algebras which is given in Example 2.3 with the derivation $\mathcal{J}$. Define the self-map $\mathcal{G}: \mathfrak{W} \longrightarrow \mathfrak{W}$ by

$$\mathcal{G}(\mathfrak{e}) = \begin{cases} 0\,, & if\ \mathfrak{e} = \zeta \\ \varsigma\,, & if\ \mathfrak{e} = 0 \\ \xi\,, & if\ \mathfrak{e} = \xi \\ \zeta, & if\ \mathfrak{e} = \varsigma \end{cases}$$

Then, it's clear that $\mathcal{G}$ is not a G-d of $\mathfrak{W}$ because $\mathcal{G}(\varsigma \circ \xi) = 0 \neq \xi = (\mathcal{G}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi))$.

Definition 2.6 Let $\mathfrak{W}$ be a BF-algebras. The self-map $\mathcal{G}$ is called regular if $\mathcal{G}(0) = 0$.

Proposition 2.7 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a g-(l, r)-d of $\mathfrak{W}$ with $\mathcal{J}$ is a regular, then $\mathcal{G}(\varsigma) = \mathcal{G}(\varsigma) \wedge \varsigma$.

Proof: Since $\mathcal{G}$ is a g-(l, r)-d of $\mathfrak{W}$, then there exist a (l, r)-d $\mathcal{J}: \mathfrak{W} \longrightarrow \mathfrak{W}$ such that $\mathcal{G}(\varsigma \circ \xi) = (\mathcal{G}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi))$ for all $\varsigma, \xi \in \mathfrak{W}$. By Definition 1.1, $\mathcal{G}(\varsigma) = \mathcal{G}(\varsigma \circ 0) = (\mathcal{G}(\varsigma) \circ 0) \wedge (\varsigma \circ \mathcal{J}(0)) = (\varsigma \circ \mathcal{J}(0)) \circ ((\varsigma \circ \mathcal{J}(0) \circ (\mathcal{G}(\varsigma) \circ 0)) = (\varsigma \circ 0) \circ ((\varsigma \circ 0) \circ (\mathcal{G}(\varsigma) \circ 0)) = \varsigma \circ (\varsigma \circ \mathcal{G}(\varsigma)) = \mathcal{G}(\varsigma) \wedge \varsigma$. ■

Proposition 2.8 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a regular g-(r, l)-d of $\mathfrak{W}$, then $\mathcal{G}(\varsigma) = \varsigma \wedge \mathcal{J}(\varsigma)$.

Proof: Since $\mathcal{G}$ is a g-(r, l)-d of $\mathfrak{W}$, then there exist a (r, l)-d $\mathcal{J}: \mathfrak{W} \longrightarrow \mathfrak{W}$ such that $\mathcal{G}(\varsigma \circ \xi) = (\varsigma \circ \mathcal{G}(\xi)) \wedge (\mathcal{J}(\varsigma) \circ \xi)$ for all $\varsigma, \xi \in \mathfrak{W}$. By Definition 1.1, we have that $\mathcal{G}(\varsigma) = \mathcal{G}(\varsigma \circ 0) = (\varsigma \circ \mathcal{G}(0)) \wedge (\mathcal{J}(\varsigma) \circ 0) = (\varsigma \circ 0) \wedge (\mathcal{J}(\varsigma) \circ 0) = \varsigma \wedge \mathcal{J}(\varsigma)$. ■

Proposition 2.9 Let $\mathfrak{W}$ be a BF-algebras with $\mathcal{G}$ is a g-(l, r)-d of $\mathfrak{W}$. Then,

i. $\mathcal{G}(\varsigma \circ \xi) = \mathcal{G}(\varsigma) \circ \xi$ for all $\varsigma, \xi \in \mathfrak{W}$,

ii. $\mathcal{G}(0) = \mathcal{G}(\varsigma) \circ \varsigma$ for all $\varsigma \in \mathfrak{W}$,

iii. $\mathcal{G}(\varsigma \circ \mathcal{J}(\varsigma)) = \mathcal{G}(\varsigma) \circ \mathcal{J}(\varsigma)$ for all $\varsigma \in \mathfrak{W}$,

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iv. If $\mathcal{G}$ is a regular, then $\mathcal{G}$ is an identity map.

Proof: (i) let $\mathcal{G}$ be a g-(l, r)-d of $\mathfrak{W}$, then $\mathcal{G}(\varsigma \circ \xi) = (\mathcal{G}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi)) = (\varsigma \circ \mathcal{J}(\xi)) \circ ((\varsigma \circ \mathcal{J}(\xi)) \circ (\mathcal{G}(\varsigma) \circ \xi)) = \mathcal{G}(\varsigma) \circ \xi$.

(ii) By Definition 1.1, $\mathcal{G}(0) = \mathcal{G}(\varsigma \circ \varsigma) = (\mathcal{G}(\varsigma) \circ \varsigma) \wedge (\varsigma \circ \mathcal{J}(\varsigma)) = (\varsigma \circ \mathcal{J}(\varsigma)) \circ ((\varsigma \circ \mathcal{J}(\varsigma)) \circ (\mathcal{G}(\varsigma) \circ \varsigma)) = \mathcal{G}(\varsigma) \circ \varsigma$. (iii) and (iv) by using the same technic. ■

Proposition 2.10 Let $\mathfrak{W}$ be a BF-algebras with $\mathcal{G}$ is a g-(r, l)-d of $\mathfrak{W}$. Then,

i. $\mathcal{G}(\varsigma \circ \xi) = \varsigma \circ \mathcal{G}(\xi)$ for all $\varsigma, \xi \in \mathfrak{W}$,

ii. $\mathcal{G}(0) = \varsigma \circ \mathcal{G}(\varsigma)$ for all $\varsigma \in \mathfrak{W}$,

iii. $\mathcal{G}(\mathcal{J}(\varsigma) \circ \varsigma) = \mathcal{J}(\varsigma) \circ \mathcal{G}(\varsigma)$ for all $\varsigma \in \mathfrak{W}$,

iv. If $\mathcal{G}$ is a regular, then $\mathcal{G}$ is an identity map.

Proof: (i) let $\mathcal{G}$ be a g-(r, l)-d of $\mathfrak{W}$, then $\mathcal{G}(\varsigma \circ \xi) = (\varsigma \circ \mathcal{G}(\xi)) \wedge (\mathcal{J}(\varsigma) \circ \xi) = (\mathcal{J}(\varsigma) \circ \xi) \circ ((\mathcal{J}(\varsigma) \circ \xi) \circ (\varsigma \circ \mathcal{G}(\xi))) = \varsigma \circ \mathcal{G}(\xi)$.

(ii) By Definition 1.1, $\mathcal{G}(0) = \mathcal{G}(\varsigma \circ \varsigma) = (\varsigma \circ \mathcal{G}(\varsigma)) \wedge (\mathcal{J}(\varsigma) \circ \varsigma) = (\mathcal{J}(\varsigma) \circ \varsigma) \circ ((\mathcal{J}(\varsigma) \circ \varsigma) \circ (\varsigma \circ \mathcal{G}(\varsigma)) = \varsigma \circ \mathcal{G}(\varsigma)$. (iii) $\mathcal{G}(\mathcal{J}(\varsigma) \circ \varsigma) = \big(\mathcal{J}(\varsigma) \circ \mathcal{G}(\varsigma)\big) \wedge \big(\mathcal{J}\big(\mathcal{J}(\varsigma)\big) \circ \varsigma\big) = (\mathcal{J}(\mathcal{J}(\varsigma)) \circ \varsigma) \circ ((\mathcal{J}(\mathcal{J}(\varsigma)) \circ \varsigma) \circ (\mathcal{J}(\varsigma) \circ \mathcal{G}(\varsigma))) = \mathcal{J}(\varsigma) \circ \mathcal{G}(\varsigma)$. (iv) Let $\mathcal{G}$ be a regular, then $\mathcal{G}(0) = 0$. By (ii) we get $\mathcal{G}(0) = \varsigma \circ \mathcal{G}(\varsigma)$ for all $\varsigma \in \mathfrak{W}$. But $\mathcal{G}$ is a regular. Thus, $\varsigma \circ \mathcal{G}(\varsigma) = 0 \Longrightarrow \mathcal{G}(\varsigma) = \varsigma$. ■

Theorem 2.11 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. Then, $\mathcal{G}$ is a regular iff $\mathcal{G}$ is an identity map.

Proof: Let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. Then, $\mathcal{G}$ is a g-(l, r)-d of $\mathfrak{W}$ and also $\mathcal{G}$ is a g-(r, l)-d of $\mathfrak{W}$. Furthermore, let $\mathcal{G}$ be a regular, then $\mathcal{G}(0) = 0$. By Proposition 3.9 (iv) and Proposition 3.10

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(iv), we have that $\mathcal{G}$ is an identity map. Conversely, let $\mathcal{G}$ be an identity map, then $\mathcal{G}(\varsigma) = \varsigma$ for every $\varsigma \in \mathfrak{W}$. By putting $\varsigma = 0$, we get $\mathcal{G}(0) = 0$ which gives that $\mathcal{G}$ is a regular. ■

Definition 2.12 Let $\mathfrak{W}$ be a BF-algebras. Furthermore, let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. We define $\mathrm{Fix}_{\mathcal{G}}(\mathfrak{W}) = \{\varsigma \in \mathfrak{W}\colon \mathcal{G}(\varsigma) = \varsigma\}$.

Proposition 2.13 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. Then, $\mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$ is a sub-algebra of $\mathfrak{W}$.

Proof: Let $\varsigma, \xi \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$ then $\mathcal{G}(\varsigma) = \varsigma$ and $\mathcal{J}(\xi) = \xi$. Since $\mathcal{G}$ is a G-d of $\mathfrak{W}$, then $\mathcal{G}$ is a g-(l, r)-d of $\mathfrak{W}$. Thereafter, $\mathcal{G}(\varsigma \circ \xi) = (\mathcal{G}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi)) = (\varsigma \circ \xi) \wedge (\varsigma \circ \xi) = \varsigma \circ \xi$. Therefore, as required. ■

Proposition 2.14 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a regular G-d of $\mathfrak{W}$. Then, the points below are holds for any $\varsigma, \xi \in \mathfrak{W}$.

i. $\mathcal{G}(\varsigma) \leq \mathcal{J}(\varsigma) \leq \varsigma$,

ii. $\mathcal{G}(\varsigma \circ \xi) \leq \varsigma \circ \mathcal{J}(\xi)$,

iii. $\mathcal{G}(\mathcal{G}(\varsigma) \circ \varsigma) = 0$,

iv. $\mathcal{G}(\mathcal{G}(\varsigma)) \leq \varsigma$.

Proof: (i) Since $\mathcal{G}$ is a G-d of $\mathfrak{W}$ then $\mathcal{G}$ is a g-(r, l)-d of $\mathfrak{W}$. Thus, by Definition 2.1, $\mathcal{G}(\varsigma) = \mathcal{G}(\varsigma \circ 0) = (\varsigma \circ \mathcal{G}(0)) \wedge (\mathcal{J}(\varsigma) \circ 0) = (\varsigma \circ 0) \wedge \mathcal{J}(\varsigma) = \varsigma \wedge \mathcal{J}(\varsigma) = \mathcal{J}(\varsigma) \circ (\mathcal{J}(\varsigma) \circ \varsigma) = \varsigma$.

(ii) $\mathcal{G}(\varsigma \circ \xi) = (\varsigma \circ \mathcal{G}(\xi)) \wedge (\mathcal{J}(\varsigma) \circ \xi) = (\mathcal{J}(\varsigma) \circ \xi) \circ ((\mathcal{J}(\varsigma) \circ \xi) \circ (\varsigma \circ \mathcal{G}(\xi))) = \varsigma \circ \mathcal{G}(\xi)$

$\leq \varsigma \circ \mathcal{J}(\xi)$ by (i). For point (iii) since $\mathcal{G}$ is a g-(r, l)-d of $\mathfrak{W}$, $\mathcal{G}(\mathcal{G}(\varsigma) \circ \varsigma) = (\mathcal{G}(\varsigma) \circ \mathcal{G}(\varsigma)) \wedge (\mathcal{J}(\mathcal{G}(\varsigma)) \circ \varsigma) = 0 \wedge (\mathcal{J}(\mathcal{G}(\varsigma)) \circ \varsigma) = (\mathcal{J}(\mathcal{G}(\varsigma)) \circ \varsigma) \circ ((\mathcal{J}(\mathcal{G}(\varsigma)) \circ \varsigma) \circ 0) = 0$.

(iv) based on (i) and Definition 2.1, $\mathcal{G}(\mathcal{G}(\varsigma)) = \mathcal{G}(\varsigma \wedge \mathcal{J}(\varsigma)) = \mathcal{G}(\mathcal{J}(\varsigma) \circ (\mathcal{J}(\varsigma) \circ \varsigma))$. Since $\mathcal{G}$ is a g-(r, l)-d of $\mathfrak{W}$, then $\mathcal{G}(\mathcal{J}(\varsigma) \circ (\mathcal{J}(\varsigma) \circ \varsigma)) = (\mathcal{J}(\varsigma) \circ \mathcal{G}(\mathcal{J}(\varsigma) \circ \varsigma)) \wedge (\mathcal{J}(\mathcal{J}(\varsigma)) \circ (\mathcal{J}(\varsigma) \circ \varsigma))$

$= (\mathcal{J}(\varsigma)\circ\mathcal{G}(0)) \wedge (\mathcal{J}(\mathcal{J}(\varsigma))\circ 0) = (\mathcal{J}(\varsigma)\circ 0) \wedge \mathcal{J}(\mathcal{J}(\varsigma)) = \mathcal{J}(\varsigma) \wedge \mathcal{J}(\mathcal{J}(\varsigma)) = \mathcal{J}(\mathcal{J}(\varsigma))\circ(\mathcal{J}(\mathcal{J}(\varsigma))\circ\mathcal{J}(\varsigma)) = \mathcal{J}(\varsigma) \leq \varsigma$. ■

Proposition 2.15 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. If $\varsigma \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$, then $\mathcal{J}(\varsigma) = \varsigma$.

Proof: Since $\varsigma \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$ then $\mathcal{G}(\varsigma) = \varsigma$. By Proposition 3.14 (i), we have $\mathcal{G}(\varsigma) \leq \mathcal{J}(\varsigma) \leq \varsigma$.

■

Proposition 2.16 Let $\mathfrak{W}$ be a BF-algebras and $\mathcal{G}$ be a G-d of $\mathfrak{W}$. If $\varsigma, \xi \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$, then $\varsigma \wedge \xi \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$.

Proof: By Proposition 2.13, $\mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$ is a sub-algebra of $\mathfrak{W}$. That is mean $\varsigma\circ\xi \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$ for any $\varsigma, \xi \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$. Since $\mathcal{G}$ is a g-(l, r)-d of $\mathfrak{W}$, then $\mathcal{G}(\varsigma \wedge \xi) = \mathcal{G}(\xi\circ(\xi\circ\varsigma)) = (\mathcal{G}(\xi)\circ(\xi\circ\varsigma)) \wedge (\xi\circ\mathcal{J}(\xi\circ\varsigma)) = (\xi\circ(\xi\circ\varsigma)) \wedge (\xi\circ(\xi\circ\varsigma)) = \xi\circ(\xi\circ\varsigma) = \varsigma \wedge \xi$. Therefore, as required. ■

Definition 2.17 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. We define the kernel of $\mathcal{G}$ by $\mathrm{Ker}\mathcal{G} = \{\varsigma \in \mathfrak{W}\colon\ \mathcal{G}(\varsigma) = 0\}$.

Example 2.18 Let $\mathcal{G}$ be a G-d defined in Example 3.3, then $\mathrm{Ker}\mathcal{G} = \{\zeta\}$.

Proposition 2.19 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. Then, $\mathrm{Ker}\mathcal{G}$ is a sub-algebra of $\mathfrak{W}$.

Proof: To show that kernel of $\mathcal{G}$ is a sub-algebra of $\mathfrak{W}$, we have to prove $\varsigma\circ\xi \in \mathrm{Ker}\mathcal{G}$ for any $\varsigma, \xi \in \mathrm{Ker}\mathcal{G}$. Now, let $\varsigma, \xi \in \mathrm{Ker}\mathcal{G}$, then $\mathcal{G}(\varsigma\circ\xi) = (\mathcal{G}(\varsigma)\circ\xi) \wedge (\varsigma\circ\mathcal{J}(\xi)) = (0\circ\xi) \wedge (\varsigma\circ 0) = 0 \wedge \varsigma = \varsigma\circ(\varsigma\circ 0) = 0$. Therefore, as required. ■

Proposition 2.20 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. If $\xi \in \mathrm{Ker}\mathcal{G}$ with $\varsigma \in \mathfrak{W}$ then $\varsigma \wedge \xi \in \mathrm{Ker}\mathcal{G}$.

Proof: Since $\xi \in \mathrm{Ker}\mathcal{G}$ then $\mathcal{G}(\xi) = 0$. Thus, $\mathcal{G}(\varsigma \wedge \xi) = \mathcal{G}(\xi\circ(\xi\circ\varsigma)) = (\mathcal{G}(\xi)\circ(\xi\circ\varsigma))$

$\wedge (\xi \circ \mathcal{J}(\xi \circ \varsigma)) = (0 \circ (\xi \circ \varsigma)) \wedge (\xi \circ \mathcal{J}(\xi \circ \varsigma)) = 0 \wedge (\xi \circ \mathcal{J}(\xi \circ \varsigma)) = (\xi \circ \mathcal{J}(\xi \circ \varsigma)) \circ$

$((\xi \circ \mathcal{J}(\xi \circ \varsigma)) \circ 0) = 0$.Therefore, $\varsigma \wedge \xi \in \mathrm{Ker}\mathcal{G}$. ■

Proposition 2.21 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be a G-d of $\mathfrak{W}$. If $\varsigma \in \mathrm{Ker}\mathcal{G}$, then $\varsigma \circ \xi \in \mathrm{Ker}\mathcal{G}$.

Proof: Since $\varsigma \in \mathrm{Ker}\mathcal{G}$ then $\mathcal{G}(\varsigma) = 0$. Now, $\mathcal{G}(\varsigma \circ \xi) = (\mathcal{G}(\varsigma) \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi)) = (0 \circ \xi) \wedge (\varsigma \circ \mathcal{J}(\xi)) = 0 \wedge (\varsigma \circ \mathcal{J}(\xi)) = (\varsigma \circ \mathcal{J}(\xi)) \circ ((\varsigma \circ \mathcal{J}(\xi)) \circ 0) = 0$. Hence, $\varsigma \circ \xi \in \mathrm{Ker}\mathcal{G}$. ■

Definition 2.22 Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two G-d of a BF-algebras $\mathfrak{W}$. We define the composition of $\mathcal{G}_1$ and $\mathcal{G}_2$ as a self-map $(\mathcal{G}_1 \circ \mathcal{G}_2)\colon \mathfrak{W} \longrightarrow \mathfrak{W}$ such that $(\mathcal{G}_1 \circ \mathcal{G}_2)(\varsigma) = \mathcal{G}_1(\mathcal{G}_2(\varsigma))$ for any $\varsigma \in \mathfrak{W}$.

Proposition 2.23 Let $\mathfrak{W}$ be a BF-algebras and $\mathcal{G}$ be a G-d of $\mathfrak{W}$. If $\varsigma \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$, then $(\mathcal{G} \circ \mathcal{G})(\varsigma) = \varsigma$.

Proof: Since $\varsigma \in \mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$ then $\mathcal{G}(\varsigma) = \varsigma$. Thus, by Definition 2.22, $(\mathcal{G} \circ \mathcal{G})(\varsigma) = \mathcal{G}(\mathcal{G}(\varsigma)) = \mathcal{G}(\varsigma) = \varsigma$. Therefore, as required. ■

Definition 2.24 Let $\mathfrak{W}$ be a BF-algebras. A self-map $\mathcal{G}\colon \mathfrak{W} \longrightarrow \mathfrak{W}$ is called an isotone if $\varsigma \leq \xi \Longrightarrow \mathcal{G}(\varsigma) \leq \mathcal{G}(\xi)$.

Proposition 2.25 Let $\mathfrak{W}$ be a BF-algebras and let $\mathcal{G}$ be G-d of $\mathfrak{W}$. If $\mathcal{G}$ is an isotone with $\varsigma \leq \xi$ and $\varsigma \in \mathrm{Ker}\mathcal{G}$ then $\xi \in \mathrm{Ker}\mathcal{G}$.

Proof: Let $\varsigma \leq \xi$ and $\varsigma \in \mathrm{Ker}\mathcal{G}$ then $\mathcal{G}(\varsigma) = 0$. Since $\mathcal{G}$ is an isotone, then $0 = \mathcal{G}(\varsigma) \leq \mathcal{G}(\xi)$ which implies that $0 = \mathcal{G}(\xi)$. Therefore, $\xi \in \mathrm{Ker}\mathcal{G}$. ■

Definition 2.26 Let $\mathfrak{W}$ be a BF-algebras and $\mathcal{G}$ be G-d of $\mathfrak{W}$. An ideal $\mathfrak{T}$ of $\mathfrak{W}$ is called $\mathcal{G}$-invariant if $\mathcal{G}(\mathfrak{T}) \subseteq \mathfrak{T}$.

Definition 2.27 Let $\mathfrak{W}$ be a BF-algebras. Then, $\mathfrak{W}$ is called torsion free BF-algebra, if for every $\varsigma \in \mathfrak{W}$ we have $\varsigma + \varsigma = \varsigma \circ (0 \circ \varsigma) = 0 \Longrightarrow \varsigma = 0$. If there exist a non-zero element $\varsigma \in \mathfrak{W}$ with $\varsigma + \varsigma = 0$, then $\mathfrak{W}$ is not torsion free BF-algebra.

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Example 2.28 (i) Any BF-algebras is not torsion free BF-algebra.

(ii) A BF-algebras which is given in Example 3.3 is not torsion free BF-algebra, since $\zeta + \zeta = \zeta \circ (0 \circ \zeta) = 0$.

(iii) Let $\mathfrak{W} = \{0,1,2\}$ is a BF-algebras given in the table below

$\circ$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Clearly that $\mathfrak{W}$ is not torsion free BF-algebra because $1 + 1 = 1 \circ (0 \circ 1) = 0$.

(iv) Let $\mathfrak{W} = \{0,1,2\}$ is a BF-algebras given in the table below

$\circ$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

Clearly that $\mathfrak{W}$ is not torsion free BF-algebra because $2 + 2 = 2 \circ (0 \circ 2) = 0$.

Conclusion

The concept of generalized derivations of BF-algebras has been introduced in this paper. Moreover, some properties of g-(l, r)-d (resp. g-(r, l)-d) of a BF-algebras have been proved. Also, we showed that the kernel of $\mathcal{G}$ is a sub-algebra of a BF-algebras. We then proved some other properties related to the kernel of $\mathcal{G}$ and $\mathrm{Fix}_{\mathcal{G}}(\mathfrak{W})$. The obtained results shows that any BF-algebras is not torsion free BF-algebra. For further research, we suggest the study of generalized (σ, τ)-derivations of BF-algebras.

References

1. N. O. Al-Shehrie. 2010. Derivation of B-Algebras. Journal of King Abdulaziz Uinvrsty Science, vol.22, no. 1, pp. 71-83.
2. K. Sawika, R. Intasan, A. Kaewwasri and A. Iampan. 2016. Derivation of UP-Algebras. Korean Journal of Math, vol. 24, no. 3, pp. 345-367.
3. G.T. Dejen and F. G. Tegegne. 2020. Derivations of BF-Algebras. International Journal of Innovative Science and Research Technology, vol. 5, no. 10, pp.329-333.
4. M. A. Ozturk, Y. Ceven and Y. B. Jun. 2009. Generalized Derivations of BCI-Algebras. Honam Mathematics Journal, vol. 31, no. 4, pp. 601-609.
5. G. Muhiuddin and A. M. Al-roqi. 2014. On Generalized Left Derivations in BCI-Algebras. Applied Mathematics and Information Sciences, vol.8, no. 3, pp. 1153-1158.
6. K. H. Kim. 2014. On Generalized Derivations of BE-Algebras. Journal of the Chungcheong Mathematical society, vol. 27, no. 2, pp. 227-236.
7. S.M. Bawazeer, N. O. Alshehri and R. S. Babusail. 2013. Generalized Derivations of BCC-Algebras. International Journal of Mathematics and Mathematical Sciences, vol. 2013, pp.1-4.
8. T. Ganeshkumar and M. Chandramouleeswaran. 2013. Generalized Derivation of TM-Algebras. International Journal of Algebra, vol. 7, no. 6,pp. 251-258.
9. R. M. Al-omary, M. S. Khan and N.U. Rehman. On Generalized Derivations in d-Algebras. 2015. Journal of Advanced Research in Pure Mathematics, vol. 7, no. 3, pp. 23-34.
10. A. Walendziak. 2007. On-BF-Algebras. Mathematica Slovaca, vol. 57, no. 2, pp. 119-128.