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Estimating the Parameters and Reliability Functions of a Weighted Probability Distribution

Duha K. Mahmoud[1*], Inam A.R. Noaman[2], Fatima M. Aboud[1]

[1]Department of Mathematics, College of Science, University of Diyala

[2]Department of statistics, College Administration and Economics, University of Diyala

[*]duhamath1999@gmail.com

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Abstract

This research is concerned with studying the estimation of the parameters and reliability function of a weighted probability distribution consisting of a combination of the Pareto distribution and the Poisson distribution. It is called the weighted Pareto-Poisson distribution, which is a new probability distribution for life time. The weighted distribution Pareto-Poisson is distribution with four parameters $\alpha, \theta, \lambda, \gamma$. In the research, the parameters and distribution reliability function were estimated in this research using two methods, namely the maximum likelihood method and the moments method. The mean square error measurement was used to compare the results of the estimates obtained from the two methods, in addition to give three sets of initial default values for the distribution parameters $\alpha, \theta, \lambda, \gamma$ and three sizes for the samples used in the estimation 30, 60, 90. The sizes are considered small, medium, and large. The process was repeated. 1000 times. In addition, five failure times were taken to estimate the reliability function. The Python programming language was used to simulate this data distribution.

Keywords: Weighted Pareto – Poisson distribution, Maximum Likelihood Method, Moment Method, and Simulation.

Introduction

The Pareto-Poisson distribution, which was presented by Ahmed El-Shahhat in 2022, has wide importance in medical, industrial, engineering, biological, and other applications for testing life time [1]. This distribution has the probability function pdf and cumulative distribution cdf are given by equations (1) and (2):

$$f(\mathrm{x}, \alpha, \lambda, \theta) = \frac{\alpha\theta\lambda^{\alpha} e^{\theta}}{\mathrm{x}^{\alpha+1}\,(e^{\theta}-1)} \qquad \alpha\,, \theta > 0\,, x \geq \lambda \quad \ldots\ldots\ldots \qquad (1)$$

$$F(\mathrm{x}, \alpha, \theta, \lambda) = 1 - \frac{e^{\theta} - e^{\theta\left(\frac{\lambda^{\alpha}}{\mathrm{x}^{\alpha}}\right)}}{e^{\theta}-1} \qquad \alpha, \theta, \lambda > 0\,, x \geq \lambda \quad, \text{x is random varaible} \ \ldots\ldots \qquad (2)$$

The researcher made some modification and additions to produce a new distribution called the weighted Pareto-Poisson distribution using the Azzallini's method which is a new way to find weighted distributions [2] [3], which is as follows, as in equation (3):

$$\mathrm{f_w}(x) = \frac{1}{\Pr(\mathrm{x}_1 < \gamma \mathrm{x}_2)} f(\mathrm{x}_1).F(\gamma \mathrm{x}_2) \ldots\ldots \qquad (3)$$

$\mathrm{x}_1\, and\ \mathrm{x}_2$ is random varaible

Where the probability density function of the Weighted Pareto – Poisson distribution is:

$$f(\mathrm{x}, \alpha, \theta, \lambda, \gamma) = \frac{(1+\gamma^{-\alpha})\,\alpha\theta\lambda^{\alpha} e^{\theta\lambda^{\alpha}\mathrm{x}^{-\alpha}}\left[e^{\theta} - e^{\theta\lambda^{\alpha}(\gamma \mathrm{x})^{-\alpha}}\right]}{\left[(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}\right]\mathrm{x}^{\alpha+1}}, \theta > 0, x \geq \lambda \geq 1, \alpha > 0 \ldots (4)$$

As for the cumulative distribution function and the reliability function are:

$$F(\mathrm{x}, \alpha, \theta, \lambda, \gamma) = 1 - \frac{(1+\gamma^{-\alpha})\,e^{\theta+\theta\lambda^{\alpha}\mathrm{x}^{-\alpha}} - e^{(1+\gamma^{-\alpha})\theta\lambda^{\alpha}\mathrm{x}^{-\alpha}}}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}} \ , \theta > 0\,,\ x \geq \lambda \geq 1\ ,\ \alpha > 0 \ldots\ldots\ldots (4)$$

$$R(x,\alpha,\theta,\lambda,\gamma) = \frac{(1+\gamma^{-\alpha})\, e^{\theta+\theta\lambda^{\alpha}x^{-\alpha}} - e^{(1+\gamma^{-\alpha})\theta\lambda^{\alpha}x^{-\alpha}}}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\ , \theta > 0\ , x \geq \lambda \geq 1\ , \alpha > 0 \dots\dots\dots (5)$$

$$E(X^r) = \frac{(1+\gamma^{-\alpha})\, \alpha^2\, \theta^{\alpha^{-1}r+1}\lambda^r}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(\alpha r - \alpha^2\ , -\theta)\right) - (1+\gamma^{-1})^{r-\alpha}\left(\Gamma(\alpha r - \alpha^2\ , -\theta - \theta\gamma^{-\alpha})\right)\right].\ \ r \in Z^+ \dots\dots (6)$$

The Theoretical Side

In this part, we will stady about the methods used to estimate the parameters of the above-mentioned distribution, which are as follows:

1- **The Maximum Likelihood Method** [4]

The researcher used the Maximum Likelihood Method to estimate distribution parameters because this method is considered one of the best methods for estimating parameters because it has characteristics such as consistency, efficiency, and others, and this method places the function at its maximum. If we have a random sample (x_1, x_2, …., x_n) from a weighted (Pareto-Poisson) distribution, this function is as follows:

$$L(X, X_2, \dots, X_n, \alpha, \theta, \lambda, \gamma) = \prod_{i=1}^{n} f(X_i, \alpha, \theta, \lambda, \gamma)$$

which can be written as follows:

$$L(X_1, X_2, \dots, X_n, \alpha, \theta, \lambda, \gamma) = \prod_{i=1}^{n} \frac{(1+\gamma^{-\alpha})\, \alpha\theta\lambda^{\alpha}\, e^{\frac{\theta\lambda^{\alpha}}{X_i{}^{\alpha}}}\left(e^{\theta} - e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}\right)}{\left((1+\gamma^{-\alpha})\, e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}\right) X_i^{\alpha+1}}$$

$$= \left(\frac{(1+\gamma^{-\alpha})\, \alpha\theta\lambda^{\alpha}}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\frac{\theta}{\gamma^{\alpha}}}}\right)^{n} e^{\sum_{i=1}^{n}\theta\left(\frac{\lambda}{X_i}\right)^{\alpha}} \prod_{i=1}^{n} X_i^{-(\alpha+1)} \prod_{i=1}^{n} f\left(e^{\theta} - e^{\theta\left(\frac{\lambda}{\gamma X_i}\right)^{\alpha}}\right) \dots\dots (7)$$

Taking the ln of both sides the equation (7), it becomes:

$$\ln(L) = n\ln(1+\gamma^{-\alpha}) + n\ln\alpha + n\ln\theta + n\alpha\ln\lambda - \frac{\partial L}{\partial\theta} n\ln\left[(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\frac{\theta}{\gamma^{\alpha}}}\right] +$$

$$\sum_{i=1}^{n}\theta\left(\frac{\lambda}{x_i}\right)^{\alpha} + \sum_{i=1}^{n}\ln\left[x_i^{-(\alpha+1)}\right] + \sum_{i=1}^{n}\ln\left[e^{\theta} - e^{\theta\left(\frac{\lambda}{\gamma x_i}\right)^{\alpha}}\right] \ldots\ldots (8)$$

By differentiating both sides of the equation (8) with respect to each α, θ, λ, γ respectively and setting it equal to zero, it becomes:

$$\frac{\partial L}{\partial\alpha} = \frac{-n\ln\gamma(\gamma^{-\alpha})}{\gamma^{\alpha}+1} + \frac{n}{\alpha} + n\ln\lambda - n\left[\frac{-\ln(\gamma)e^{\theta}+\ln(\gamma)e^{\frac{\theta}{\gamma^{\alpha}}}}{\gamma^{\alpha}e^{\theta}+e^{\theta}+\gamma^{\alpha}e^{\frac{\theta}{\gamma^{\alpha}}}}\right] + \sum_{i=1}^{n}\frac{\theta\lambda^{\alpha}(\ln\lambda-\ln X_i)}{X_i^{\alpha}} - \sum_{i=1}^{n}\ln X_i +$$

$$\sum_{i=1}^{n}\frac{\theta\left(\ln(\gamma)+\ln(X_i)e^{\frac{\theta}{(\gamma X_i)^{\alpha}}}\right)}{\gamma X_i{}^{\alpha}\left(e^{\theta}-e^{\frac{\theta}{(\gamma X_i)^{\alpha}}}\right)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\frac{\partial L}{\partial\theta} = \frac{n}{\theta} - \frac{n(1+\gamma^{-\alpha})\left[(1+\gamma^{-\alpha})e^{2\theta}-e^{\theta+\frac{\theta}{\gamma^{\alpha}}}\right]}{\left[(1+\gamma^{-\alpha})e^{2\theta}-e^{\theta+\frac{\theta}{\gamma^{\alpha}}}\right]} + \sum_{i=1}^{n}\left(\frac{\lambda}{X_i}\right)^{\alpha} + \sum_{i=1}^{n}\frac{e^{\theta}-\lambda^{\alpha}e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}}{(\gamma X_i)^{\alpha}\left(e^{\theta}-e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}\right)} = 0 \quad \ldots\ldots (10)$$

$$\frac{\partial L}{\partial\lambda} = \frac{n\alpha}{\lambda} + \sum_{i=1}^{n}\frac{\theta\alpha\lambda^{\alpha-1}}{X_i^{\alpha}} - \sum_{i=1}^{n}\frac{\alpha\theta\lambda^{\alpha-1}\,e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}}{(\gamma X_i)^{\alpha}\left(e^{\theta}-e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}\right)} = 0 \quad \ldots\ldots (11)$$

$$\frac{\partial L}{\partial\gamma} = \frac{-n\alpha}{(\gamma^{\alpha}+1)\gamma} + \frac{n\alpha\gamma^{-(\alpha+1)}e^{\theta}\left(e^{\theta}-\theta e^{\frac{\theta}{\gamma^{\alpha}}}\right)}{e^{\theta}(1+\gamma^{-\alpha})e^{\theta}-e^{\frac{\theta}{\gamma^{\alpha}}}} + \sum_{i=1}^{n}\frac{\alpha\theta\lambda^{\alpha}\,e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}}{\gamma^{(\alpha+1)}\left(x^{\alpha}e^{\theta}-x^{\alpha}e^{\frac{\theta\lambda^{\alpha}}{(\gamma X_i)^{\alpha}}}\right)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

After derivation and equating the equations 9, 10, 11, 12 with zero, non-linear equations appeared that cannot be solved Newton-Raphson method to find the estimated parameters After finding the values of the estimators, these values are substituted into the equation (5) in order to obtain a Maximum Likelihood Estimate of the reliability function of the distribution.

$$\hat{R}(x,\hat{\alpha},\hat{\theta},\hat{\lambda},\hat{\gamma}) = \frac{(1+\hat{\gamma}^{-\hat{\alpha}})\, e^{\hat{\theta}+\hat{\theta}\hat{\lambda}^{\hat{\alpha}}x^{-\hat{\alpha}}} - e^{(1+\hat{\gamma}^{-\hat{\alpha}})\hat{\theta}\hat{\lambda}^{\hat{\alpha}}x^{-\hat{\alpha}}}}{(1+\hat{\gamma}^{-\hat{\alpha}})e^{2\hat{\theta}} - e^{\hat{\theta}+\hat{\theta}\hat{\gamma}^{-\hat{\alpha}}}} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (13)$$

2- **Estimation Using the Method of Moments** [5]

The method of moments is one of the most widely used methods for estimating parameters as follows:

$$m_r = \mu_r \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..(14)$$

Where

$$\mu_r = E(X^r) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(15)$$

$$m_r = \frac{\sum_{i=1}^{n} x_i^r}{n} \quad \text{for } r \in Z^+ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots ..(16)$$

Since

$$E(X^r) = \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{\alpha^{-1}r+1}\lambda^r}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(\alpha r - \alpha^2, -\theta)\right) - (1+\gamma^{-1})^{r-\alpha}\left(\Gamma(\alpha r - \alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(17)$$

Then By the equation (14) it is obtained:

$$\frac{\sum_{i=1}^{n} x_i^r}{n} = \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{\alpha^{-1}r+1}\lambda^r}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(\alpha r - \alpha^2, -\theta)\right) - (1+\gamma^{-1})^{r-\alpha}\left(\Gamma(\alpha r - \alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right]$$

$$\frac{\sum_{i=1}^{n} x_i^r}{n} - \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{\alpha^{-1}r+1}\lambda^r}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(\alpha r - \alpha^2, -\theta)\right) - (1+\gamma^{-1})^{r-\alpha}\left(\Gamma(\alpha r - \alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right] = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (18)$$

When $r = 1,\ m_1 = \mu_1$

$$\frac{\sum_{i=1}^{n} x_i}{n} - \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{\alpha^{-1}+1}\lambda}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(\alpha - \alpha^2, -\theta)\right) - (1+\gamma^{-1})^{1-\alpha}\left(\Gamma(\alpha - \alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right] = 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (19)$$

When $r = 2,\ m_2 = \mu_2$

$$\frac{\sum_{i=1}^{n} X_i^2}{n} - \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{2\alpha^{-1}+1}\lambda^2}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(2\alpha-\alpha^2, -\theta)\right) - (1+\gamma^{-1})^{2-\alpha}\left(\Gamma(2\alpha-\alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right] = 0 \quad \ldots (20)$$

When $r = 3,\ m_3 = \mu_3$

$$\frac{\sum_{i=1}^{n} X_i^3}{n} - \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{3\alpha^{-1}+1}\lambda^3}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(3\alpha-\alpha^2, -\theta)\right) - (1+\gamma^{-1})^{3-\alpha}\left(\Gamma(3\alpha-\alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right] = 0 \quad \ldots (21)$$

When $r = 4,\ m_4 = \mu_4$

$$\frac{\sum_{i=1}^{n} X_i^4}{n} - \frac{(1+\gamma^{-\alpha})\,\alpha^2\,\theta^{4\alpha^{-1}+1}\lambda^4}{(1+\gamma^{-\alpha})e^{2\theta} - e^{\theta+\theta\gamma^{-\alpha}}}\left[e^{\theta}\left(\Gamma(4\alpha-\alpha^2, -\theta)\right) - (1+\gamma^{-1})^{4-\alpha}\left(\Gamma(4\alpha-\alpha^2, -\theta - \theta\gamma^{-\alpha})\right)\right] = 0 \quad \ldots (22)$$

After derivation and equating the equations 8, 19, 20, 21 with zero, non-linear equations appeared that can be solved by Newton-Raphson method was used to find the estimated parameters. After finding the values of the estimators, these values are substituted into the equation (5) in order to obtain a Moment Estimate of the reliability function of the distribution.

Experimental Aspect

In this aspect, simulation was conducted in order to compare the estimator results of the methods used in estimation, and the simulation steps were as follows:

- Choose default values for complex distribution parameters $(\alpha, \theta, \lambda, \gamma)$, Table 1 shows these values.

Table 1: Default values for parameters

α	θ	λ	γ
0.5	0.8	0.1	0.5
0.5	0.8	0.2	0.6
0.5	0.4	0.1	0.5

- Three different sizes were chosen (small=30, medium=60, large=90) repeat the process 1000 times.
- Choose five failure times in order to estimate the reliability function.
- Use the gamma distribution to generate random data.
- Use the Mean Square Error (MSE) measure to compare and obtain the best estimator

$$MSE = \frac{1}{R}\sum_{i=1}^{R}(\varphi - \hat{\varphi})^2$$

Where

φ: The true value of the parameter.

$\hat{\varphi}$: Estimated value of the parameter.

R: Number of repetitions.

Result Simulation of Analysis

1- Parameter estimation

The results of the estimated parameters of the simulation experiment, which were written using a program written in Python, were presented and analyzed according to the following tables (2, 3, 4):

Table 2: Estimated values and MSE values for the estimation methods used When $\alpha = 0.5, \theta = 0.8, \lambda = 0.1, \gamma = 0.5$

Parameters			MLE	Moments
n=30	α	$\hat{\alpha}$	0.5000003	1.6008985685
		MSE	8.99999999e-17	0.0012119776
	θ	$\hat{\theta}$	0.7976541	0.2521967227
		MSE	5.50324681e-09	0.0003000884
	λ	$\hat{\lambda}$	0.0901474	0.8797024262
		MSE	9.70737267e-08	0.0006079358
	γ	$\hat{\gamma}$	0.5226594	1.3840591321
		MSE	5.13448408e-07	0.0007815605
n=60	α	$\hat{\alpha}$	0.50000000007	0.1756937191
		MSE	4.90000081e-24	0.0001051745
	θ	$\hat{\theta}$	0.7999900017	1.7260555766
		MSE	9.99660028e-14	0.0008575789
	λ	$\hat{\lambda}$	0.1999800027	0.2608780876
		MSE	9.99600093e-06	2.58817590e-05
	γ	$\hat{\gamma}$	0.6001072337	0.9887307727
		MSE	1.00214582e-05	0.0002388577
n=90	α	$\hat{\alpha}$	0.5000000005	0.6482610772
		MSE	2.50000041e-22	2.19813470e-05
	θ	$\hat{\theta}$	0.79998417	0.8195751728
		MSE	2.50588900e-13	3.831873901e-07
	λ	$\hat{\lambda}$	0.09993667	0.5622135973
		MSE	4.01068890e-12	0.0002136414
	γ	$\hat{\gamma}$	0.50016542	2.5969820800
		MSE	2.73637763e-11	0.0043973338

Table 3: Estimated values and MSE values for the estimation methods used When $\alpha = 0.5, \theta = 0.8, \lambda = 0.2, \gamma = 0.6$

Parameters			MLE	Moments
n=30	α	$\hat{\alpha}$	0.500000000007	2.0708223514
		MSE	4.89993863e-24	0.0024674828
	θ	$\hat{\theta}$	0.79999953898	0.1771488922
		MSE	2.12539440e-16	0.00038794350
	λ	$\hat{\lambda}$	0.19999907797	1.0800356286
		MSE	8.50139320e-16	0.0007744627
	γ	$\hat{\gamma}$	0.60000473196	1.6891589598
		MSE	2.23914454e-14	0.0011862672
n=60	α	$\hat{\alpha}$	0.500000000002	0.0451693191
		MSE	3.99982302e-27	0.0002068709
	θ	$\hat{\theta}$	0.79999999929	0.2378681119
		MSE	5.04100083e-22	0.0003159922
	λ	$\hat{\lambda}$	0.19999999859	0.3923376479
		MSE	1.98810001e-21	3.69937707e-05
	γ	$\hat{\gamma}$	0.60000000653	1.6708318995
		MSE	4.26408998e-20	0.0011466809
n=90	α	$\hat{\alpha}$	0.50000000032	1.0382462819
		MSE	1.02400016e-22	0.0002897090
	θ	$\hat{\theta}$	0.79991754099	2.0506693592
		MSE	6.79948833e-12	0.0015641738
	λ	$\hat{\lambda}$	0.19983502250	0.5651069417
		MSE	2.72175755e-11	0.0001333030
	γ	$\hat{\gamma}$	0.60091057049	2.3222399709
		MSE	8.29138617e-10	0.0029661105

Table 4: Estimated values and MSE values for the estimation methods used When $\alpha = 0.5,\ \theta = 0.4,\ \lambda = 0.1,\ \gamma = 0.5$

Parameters			MLE	Moments
n=30	α	$\hat{\alpha}$	0.5000000015	1.902852070
		MSE	2.25000003e-21	0.0019679939
	θ	$\hat{\theta}$	0.3999964356	1.163761875
		MSE	0.0001600028	0.0001323227
	λ	$\hat{\lambda}$	0.0999928711	0.067525331
		MSE	1.00014258e-05	1.75495379e-05
	γ	$\hat{\gamma}$	0.5000364877	2.014011639
		MSE	9.99270379e-06	0.0019994289
n=60	α	$\hat{\alpha}$	0.500000000001	1.8396173618
		MSE	9.99955757e-28	0.0017945746
	θ	$\hat{\theta}$	0.3999999997	0.7447267783
		MSE	0.0001600000	3.05512903e-06

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	λ	$\hat{\lambda}$	0.0999999995	0.8962792372
		MSE	1.00000001e-05	0.0004848047
	γ	$\hat{\gamma}$	0.5000000018	1.5206147879
		MSE	9.99999963e-06	0.0008475315
n=90	α	$\hat{\alpha}$	0.5000001752	1.0472279359
		MSE	3.06950399e-17	0.0002994584
	θ	$\hat{\theta}$	0.3973052504	1.3580211762
		MSE	0.0001621630	0.0003113876
	λ	$\hat{\lambda}$	0.0944764825	0.1687739618
		MSE	1.11352127e-05	9.75065461e-07
	γ	$\hat{\gamma}$	0.5305647650	2.9022535674
		MSE	4.82125185e-06	0.0053003714

From the tables (2, 3, and 4) it was noted that the Maximum likelihood method is better than the moment's method in estimating parameters because it has the lowest mean square error (MSE).

2- Estimating the reliability function [4]

In this paragraph, the reliability function of the distribution will be estimated and compared with the true value, as follows tables (5, 6, and 7):

Estimated reliability values and their MSE values for the estimation methods used

Table 5: The values when $\alpha = 0.5, \theta = 0.8, \lambda = 0.1, \gamma = 0.5$

M	t	Real	MLE	Moments
n=30	1.09	0.99850782	0.97466671	0.98104769
	MSE		0.0005683985	0.0003048561
	2.09	0.96362552	0.94153630	0.94256220
	MSE		0.0004879336	0.0004436634
	3	0.94789179	0.92671502	0.93213695
	MSE		0.0004484555	0.0002482149
	4	0.93705140	0.91653420	0.92709329
	MSE		0.0004209554	9.91639547e-05
	5	0.92957358	0.90952354	0.92447193
	MSE		0.000402004104	2.60268327e-05
n=60	1.09	0.99850782	0.95817923	0.89998652
	MSE		0.0016263951	0.0097064465
	2.09	0.96362552	0.92608118	0.83301284
	MSE		0.0014095774	0.0170596721
	3	0.94789179	0.91178903	0.79690191
	MSE		0.0013034092	0.0227979438
	4	0.93705140	0.90198917	0.76933425
	MSE		0.0012293599	0.0281290424
	5	0.92957358	0.89524783	0.74879725
	MSE		0.0011782571	0.0326800814

n=90	1.09	0.99850782	0.99833148	0.83737092
	MSE		3.10957955e-08	0.0259651005
	2.09	0.96362552	0.96346034	0.73738333
	MSE		2.72844323e-08	0.0511855285
	3	0.94789179	0.94773259	0.69939698
	MSE		2.53446400e-08	0.0617496705
	4	0.93705140	0.93689655	0.67554547
	MSE		2.39785224e-08	0.0683853514
	5	0.92957358	0.92942183	0.66015636
	MSE		2.30280625e-08	0.0725856384

The Table (5) shows us the estimated values of the reliability function and its MSE. It was noted that the estimated value of the reliability function was very close to its true value when the sample size was equal to 90, and the following graphs show the comparison between the sample sizes and the methods used. In the sample size of 90, the true value and the value estimated by the maximum likelihood method were somewhat equal, which led to the disappearance of the true value curve in Figure 3.

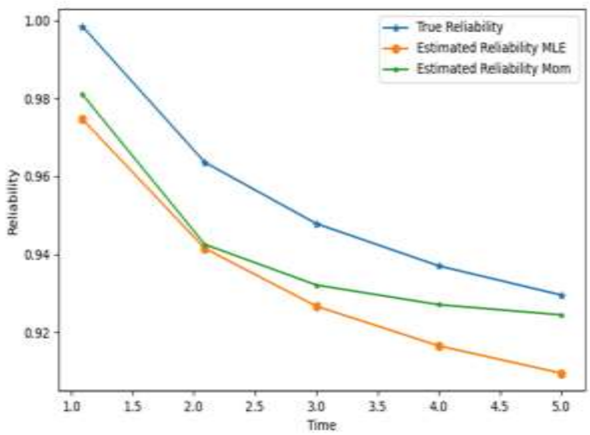

Figure 1: That Reliability Functions when Sample Size 30

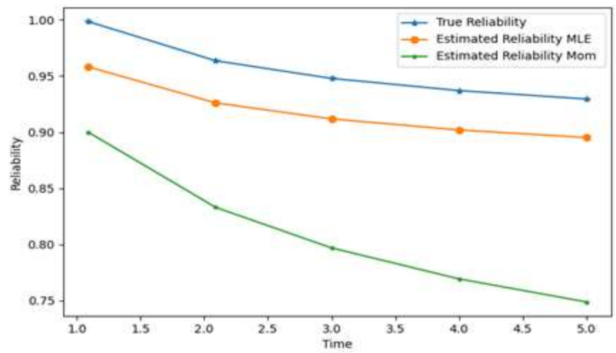

Figure 2: That Reliability Functions when Sample Size 60 Prepared by the researcher

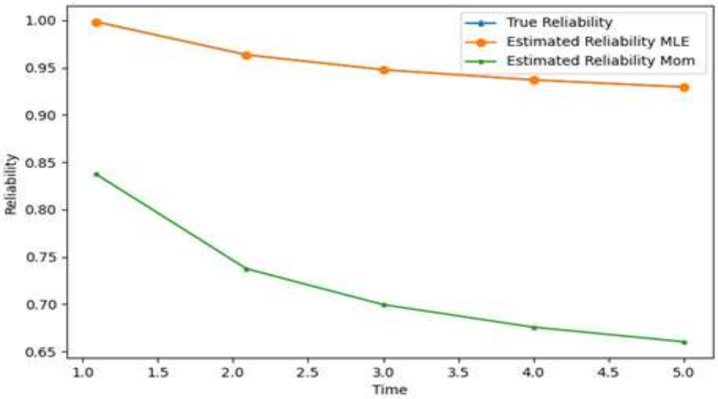

Figure 3: That Reliability Functions when Sample Size 90

Table 6: The values when $\alpha = 0.5, \theta = 0.8, \lambda = 0.2, \gamma = 0.6$

M	t	Real	MLE	Moments
n=30	1.09	0.97920504	0.97920229	0.99889475
	MSE		7.56249999e-12	0.0003876846
	2.09	0.93552150	0.93551882	0.95211233
	MSE		7.18240000e-12	0.0002752556
	3	0.91510155	0.91509893	0.94271750
	MSE		6.86440000e-12	0.0007626406
	4	0.90087741	0.90087484	0.93887804
	MSE		6.60490000e-12	0.0014440478
	5	0.89101105	0.89100852	0.93712380
	MSE		6.40090000e-12	0.0021263857
n=60	1.09	0.97920504	0.97914295	0.99977188
	MSE		3.85510601e-09	0.0004229949
	2.09	0.93552150	0.93546084	0.99952422
	MSE		3.67879853e-09	0.0040963481
	3	0.91510155	0.91504230	0.99935626
	MSE		3.50953162e-09	0.0070988561
	4	0.90087741	0.90081933	0.99920820
	MSE		3.37294954e-09	0.0096689442
	5	0.89101105	0.89095384	0.99908505
	MSE		3.27264084e-09	0.0116799894
n=90	1.09	0.97920504	0.97868029	0.39676006
	MSE		2.75362562e-07	0.3392421547
	2.09	0.93552150	0.93500847	0.24582348
	MSE		2.63199780e-07	0.4756833587
	3	0.91510155	0.91460024	0.21019416
	MSE		2.51311716e-07	0.4968944284
	4	0.90087741	0.90038575	0.19228289
	MSE		2.41729555e-07	0.5021061937
	5	0.89101105	0.89052666	0.18235642
	MSE		2.34633672e-07	0.5021913846

In the table (6), the estimated values for the reliability function and its MSE were found. It was noted that the estimated value of the reliability function was very close to its true value at all

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sample sizes. The following graphs show the comparison between the sample sizes and the methods used.

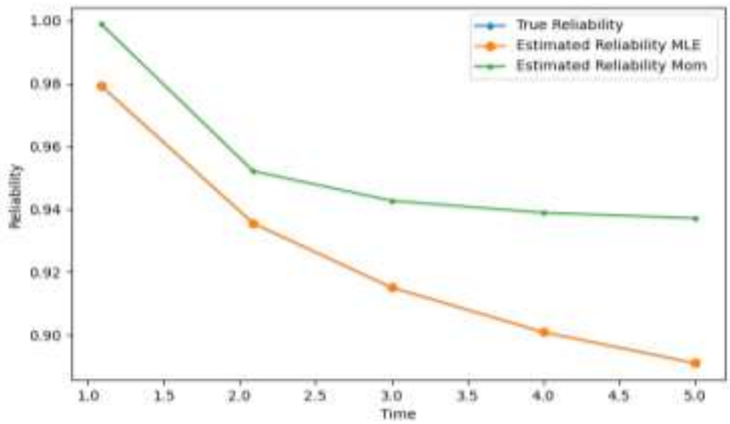

Figure 4: That Reliability Functions when Sample Size 30

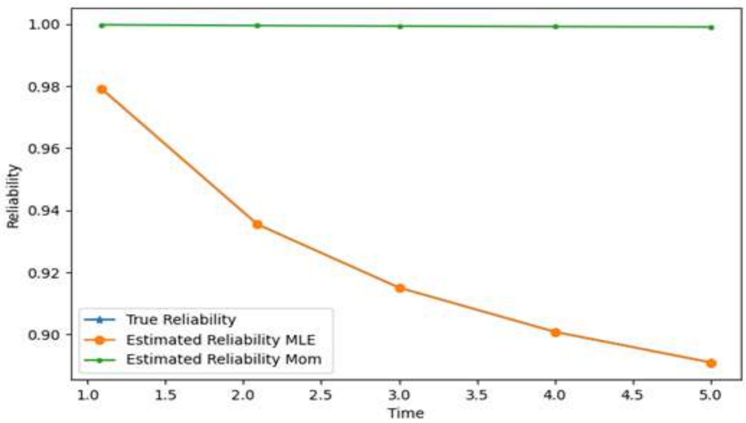

Figure 5: That Reliability Functions when Sample Size 60

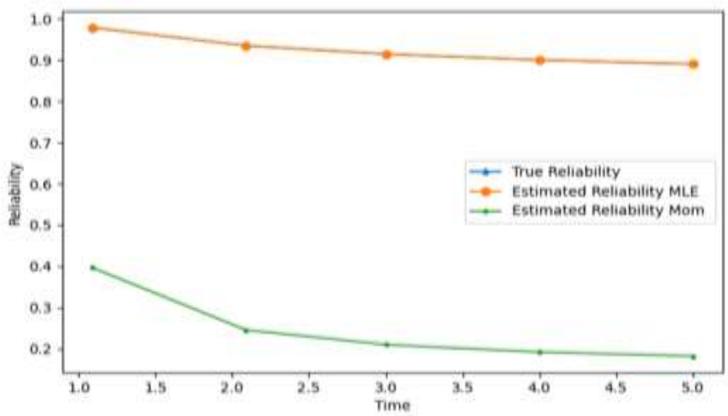

Figure 6: That Reliability Functions when Sample Size 90

Table 7: The values when $\alpha = 0.5,\ \theta = 0.8,\ \lambda = 0.2,\ \gamma = 0.6$

M	t	Real	MLE	Moments
n=30	1.09	0.99248442	0.99247650	0.35598172
	MSE		6.27264000e-11	0.4051356871
	2.09	0.98156066	0.98155293	0.35463303
	MSE		5.97528999e-11	0.3930382532
	3	0.97644055	0.97643296	0.35436001
	MSE		5.76080999e-11	0.3869841982
	4	0.97284371	0.97283624	0.35424373
	MSE		5.58009000e-11	0.3826659352
	5	0.97032975	0.97032237	0.35418854
	MSE		5.44644000e-11	0.3796299906
n=60	1.09	0.99248442	0.9924844208	0.86518519
	MSE		6.3999992827e-19	0.0162050939
	2.09	0.98156066	0.9815606685	0.66737280
	MSE		7.2250000631e-17	0.0987140113
	3	0.97644055	0.9764405506	0.62989852
	MSE		3.6000005957e-19	0.1200913785
	4	0.97284371	0.9728437162	0.61407034
	MSE		3.8439999477e-17	0.1287183310
	5	0.97032975	0.9703297575	0.60653618
	MSE		5.6249999316e-17	0.1323457615
n=90	1.09	0.99248442	0.98624409	0.35464280
	MSE		3.89417185e-05	0.4068419322
	2.09	0.98156066	0.97546895	0.32502242
	MSE		3.71089307e-05	0.4310424605
	3	0.97644055	0.97046150	0.31598857
	MSE		3.57490389e-05	0.4361968178
	4	0.97284371	0.96695587	0.31098870
	MSE		3.46666598e-05	0.4380520542
	5	0.97032975	0.96451073	0.30806113
	MSE		3.38609937e-05	0.4385997250

In the table above, the estimated values for the reliability function and its MSE were found. It was noted that the value of the reliability function estimated by the maximum likelihood method was gradually approaching the true value as the sample size increased, but the values estimated by the Moments method were moving away. When the sample size was equal to 90, there was a great convergence between them. The following two values and graphs show the comparison between the values and the methods used.

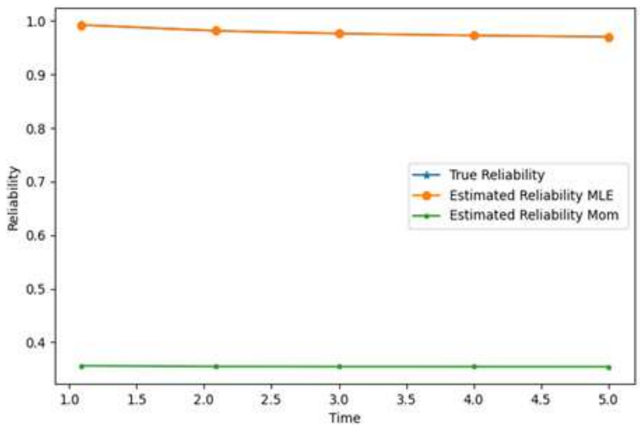

Figure 7: show that Reliability Functions when Sample Size 30

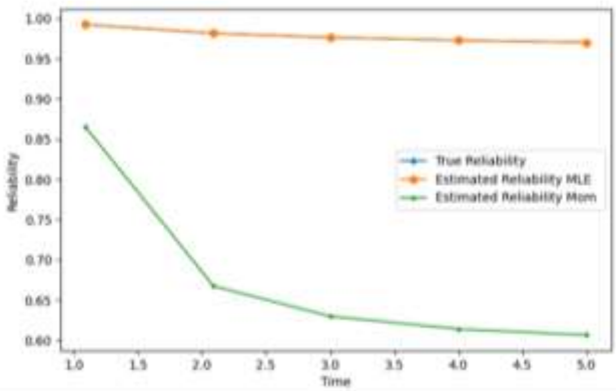

Figure 8: show that Reliability Functions when Sample Size 60

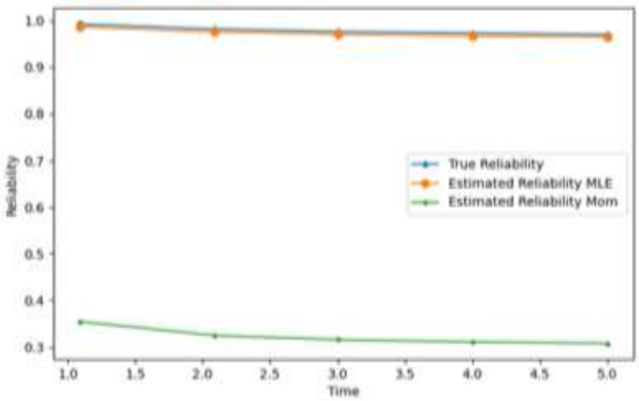

Figure 6: show that Reliability Functions when Sample Size 90

Conclusions

The most important conclusions reached by the researcher in the simulation experiments to estimate the parameters and the reliability function are:

The maximum likelihood method was the better of the two methods by which it was estimated because it has the lowest mean square error in all hypothetical data and for all sample sizes. The maximum likelihood method was the best in estimating the reliability function because it had the least mean square error in all sample sizes, except when the sample size was 30 in the data ($\alpha = 0.5, \theta = 0.8, \lambda = 0.1, \gamma = 0.5$), the moments method was the best. The value of the reliability function estimated by the maximum likelihood method got closer as the sample size increased and for all hypothetical data.

Discussion

In the previous tables, it was found that the Maximum Likelihood method is the best in estimating parameters among the two methods above, at a sample size of 60 and for the default values $\alpha = 0.5,\ \theta = 0.4,\ \lambda = 0.1,\ \gamma = 0.5$. It is also the best in estimating the reliability function when the sample size is 60, and Figure 5 shows this.

Recommendations

1- The researcher recommends using other methods to estimate the parameters of the weighted composite (Pareto-Poisson) distribution other than the methods used by the researcher, including Bayesian methods.

2- Using the weighted Pareto-Poisson distribution in other applications, especially industrial, engineering, and others

3- Use other failure data and different sample sizes

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Ethical clearance: Ethical approval was not required for this review.

References

[1] A. El-Shahhat, The Pareto–Poisson Distribution: Characteristics, Estimations and Engineering Applications, IJS, 85(1), 1-42(2022), DOI(https://doi.org/10.1007/s13171-022-00302-6)

[2] A. A. Azzallini's, class of distributions which includes the normal ones, JSTOR, 12(2), 171-178(1985)

[3] E. Q. Abdel Rahman, find better estimate of the parameters and reliability function of the compound distribution (Pareto – Truncated Exponential) with practical application, Master Thesis, Al-Basra: Al-Basra University, (2022)

[4] F. H. J. Abdul-Hassan, 'Estimating the reliability of an extended power function distribution using (NLTE-X Family) with practical application, Master Thesis, University of Karbala, Karbala, (2022)

[5] M. A. Shamran, Modified Weighted Pareto Distribution Type I (MWPDTI), Baghdad Sci J., March; 17(3), 869-877(2020),DOI(http://dx.doi.org/10.21123/bsj.2020.17.3.0869)

[6] M. W. Nehme Strucre Weighted Exponential-Pareto Distribution with Practical Application, PhD dissertation, Baghdad: Baghdad University, (2015)