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Literature Nots on Bornological Set and Bornological Group

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Abstract

In this work, we explain the basic concepts of Bornological structures Bornological sets and Bornological groups, Which solve the boundary problems for sets and groups.With some examples and fundamental construction for this structure, such as the Bornological subset, the product bornology, the Bornological isomorphism, We also explain ideal bornological . The motivation behind this part is to require less restrictive condition of boundedness correctly ideal.

Keywords: Bounded set, Bounded map, Semi bounded set, Semi bounded map, Bornological Set, Bornological Group.

Introduction

In this literature, bornological structures are considered, which are used to deal with the bounds problem in a general way see [1],[2] [3],[4],[5],[6].

1. Bornological Sets:

In this section, the bornological structure is explained with some detailed examples.

Definition (1.1) (Bornological set) [1]:

A bornology on a set X is a family $\beta \subset \mathcal{P}(X)$ such that:

(i) β covers X, i.e. $X = \bigcup_{B\in\beta} B$;

Can satisfy the covering in three ways. First, if the set X belong to the collection of bornology, then X covers itself. Second, if $\forall$ x $\in$ X, {x} $\in$ β. Third, if $X = \bigcup_{B\in\beta} B$.

(ii) β is hereditary under inclusion, i.e. if $A \subset B$ and $B \in \beta$ then $A \in \beta$;

(iii) β stable under a finite union, i.e., if B_1, $B_2 \in \beta$, then $B_1 \cup B_2 \in \beta$.

A pair (X, β) consisting of a set X and a bornology β on X is called ***a* Bornological set**, and the elements are called bounded sets.

Some types of a bornology.

i. The discrete bornology (β_{dis}) is the collection of all subsets of X, i.e. $\beta_{dis} = P(X) = 2^X$.

ii. The usual bornology (β_u) (canonical bornological set) is the collection of all usual bounded subsets of X.

iii. The finite bornology (β_{fin}) is the collection of all finite bounded subsets of X.

Definition (1.2) [1]:

A sub collection β_0 is ***a base*** if for every element of the bornology is contained in an element of the base.

Example (1.3):

Let $X = \{1,3,5\}$, there is only one bornology on this finite set which is discrete $\beta = P(x)$ the most important different between bornology and topology, is we can just defined only one bornology on finite set discrete bornology. Now, the base for this bornology

$\beta_0 = \; = \{\{1,3\}, \{1,5\}, \{3,5\}, X\}$, or $\beta_0 = \{X\}$.

Example (1.4) [6]:

Take R with (absolute value). The usual bornology on R β= {B: B ⊆ (a, b)}.(since any set in ℝ is bounded if it is obsovent in an interval).

To prove that β is bornologyon ℝ we have to satisfy three conditions

1-Since $\forall x \in \mathbb{R}$, there is an interval (a, b) such that $x \in (a, b)$, and $\{x\} \subseteq (a, b)$, then $\forall x \in \mathbb{R}$, $\{x\} \in$ β. Thus β coveringℝ;

2-If $B_1 \in$ β and $B_2 \subseteq B_1$, since ∃ bonded interval s.t$B_2 \subseteq B_1 \subseteq (a, b)$. Therefore, $B_2 \in$ β;

3- If B_1, $B_2 \in$ β, then ∃ L_1, L_2 least upper bounds and g_1, g_2 greater lower bounds.

Assume that $L = \min_{1\leq i\leq 2} \{L_i\}$ and $g = \max_{1\leq i\leq 2} \{g_i\}$

$\bigcup_{i=1,2} B_i$ Has least upper bounded L and greater lower bounded g

$\bigcup_{i=1}^{n} B_i$ Is bounded subset of R. Then, $\beta_0 = \{B_r(x): r \in R, x \in R\} = \{(x - r, x + r): r \in R, x \in R\}$.

Remarks (1.5):

Not every sub collection of a bornology forms a base.

For example $= \{1,3,5\}$, $\beta_0 = \{\{1\},\{3\},\{5\}\}$, then β_0 cannot be a base for any discrete bornology because did not achieve the base concept.

Definition (1.6) (bounded map) [1]:

A map between two bornological sets is ***bounded map*** if the image of every bounded set in its domain is bounded its codomain.

Example (1.7):

1-Let $X = \{1,2,3\}$ with $\beta = \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\}$. The identity map $f:(X,\beta) \to (X,\beta)$, It is bounded map.

2- Let $(\mathbb{R},\beta)$ be the canonical bornological set. Then, the map $f:\mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined as $f(x) = \frac{1}{x}$ is not bounded map. Indeed, $B = (0,1]$ bounded set in $\mathbb{R}\backslash\{0\}$, then the image of this bounded set under f is $[1,\infty)$ that is not bounded in $\mathbb{R}$. Therefore f is not bounded map.

Proposition (1.8):

If $f:(X,\beta) \to (X',\beta')$ and $f':(X',\beta') \to (X'',\beta'')$ are two bounded maps. Then $f \circ f'$: $(X,\beta) \to (X'',\beta'')$ is bounded.

Proof: Let $B \in \beta$ and $f:(X,\beta) \to (X',\beta')$ is bounded, thus $f(B) \in \beta'$ is bounded set bellowing to β'. Also, that f' is bounded map. It follows that $f'(f(B)) \in \beta''$. So, the composition $f' \circ f:(X,\beta') \to (X'',\beta'')$ is bounded map.

Definition (1.9) (bornological isomorphism) [1]:

A map f between two bornologicale sets is **bornological isomorphism** if it is bijective and f, f^{-1} are bounded maps.

Example (1.10):

Let $f: \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x - 1$ and a bornology on $\mathbb{R}$ is usual bornology, i.e. $f[(a,b)] = (a-1, b-1)$. It is clear that f is bijective and bounded map. Since $f(x) = y \leftrightarrow x = f^{-1}(y)$. Then, $y - 1 = x \leftrightarrow y = x + 1$.

Therefore, $f^{-1}(y) = x + 1$. So, $f^{-1}((a,b)) = (a+1, b+1)$ is bounded, then f is bornological isomorphism.

Every bornological isomorphism map is bounded which is clear from the concept of bornological isomorphism, but not every bounded map is bornological isomorphism as in the following example.

Example (1.11):

Let $f: (\mathbb{R}, \beta_u) \to (\mathbb{R}, \beta_{dis})$ defined by$f(x) = x$, $\forall\, x \in X$. The function f is bijective and bounded but f^{-1} is not bounded. Therefor f is not bornological isomorphism.

Example (1.12):

Consider $X = (-3,3)$ and $Y = (-5,7)$ with usual bornology and $f: (X, \beta_u) \to (Y, \beta_u)$ defined as $f(x) = 2x + 1$. It is clear that f is bijective and bounded map.

Then, $f^{-1}(y) = \frac{y-1}{2}$. Furthermore, f^{-1} exist and bounded map. Then, f is bornological isomorphism.

Proposition (1.13):

If f is bornological isomorphism, then f^{-1} is also bornological isomorphism.

Proof: Since f is bijective, then f^{-1} is bijective. Since f is bornological isomorphism, then f^{-1} is bounded also, $f = (f^{-1})^{-1}$ is bounded. Therefore, f^{-1} is bornological isomorphism.

Proposition (1.14):

Let(X, β) be a bornological set and $Y \subseteq X$. Then, $\beta_Y = \{V \cap Y : V \in \beta\}$ is a bornology on Y.

Proof: To prove that β_Y covers$Y$,

i) $Y = \bigcup_{B \in \beta_Y} B$

where $B = V \cap Y, V \in \beta$. Then $\bigcup_{B \in \beta_Y} B = \bigcup_{B \in \beta_Y} V \cap Y = (\bigcup_{v \in \beta}) V \cap Y$.

Where $\bigcup_{v \in \beta} V = x$ and $X \cap Y = Y$.

Then $y = \bigcup_{B \in \beta_Y} B$, then β_Y covers Y.

ii) Let $B \in \beta_Y$, i.e. $B = V \cap Y$, $V \in \beta$, then take $A \subseteq Y$, $A \subseteq B$, then $A \subseteq V \cap Y$.

$\because A \subseteq V \cap Y \longrightarrow A \subseteq V$ and $A \subseteq Y \because V \in \beta$, and β have hereditary properties, then $A \in \beta$.

To prove that $A \in \beta_Y$ we must satisfy that $A = U \cap Y$, where $U \in \beta$ .Take $A = U \therefore A = U \cap Y$ and $U \in \beta$. Then $A \in \beta_Y$.

iii) Let $\{ B_i \}_{i=1,\ldots n}$, be a finite element of β_Y , to prove β_Y is stable under finite union .

Since $\forall$ i = 1, 2 ,…, n $\exists V_i \in \beta$ such that $B_i = V_i \cap Y$

$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (V_i \cap Y) = (\bigcup_{i=1}^{n} V) \cap Y = V \cap Y$(Since β is stable under finite union).

Then $V \cap Y \in \beta_Y$.

Remark (1.15)

It is clear that every bounded subset of Y is also a bounded subset of X, i.e. $\beta_Y \subseteq \beta$.

2. Bornological Groups

In [3], when they wanted to solve the problem of bound to group they define bornology on group. In order to address the issues of boundedness for groups, the idea of bornological group are confided.

Definition (2.1) (bornological group)[8]:

A **bornological group** is a set G with two structures. G is a ***group,*** and (G, β) is a **bornological set**, such that, these two structures are compatible and.

(i) The product function $f : G \times G \to G$ is bounded;

(ii) The inverse function $f^{-1} : G \to G^{-1}$ is bounded.

Let G be a ***bornological group*** and B_1 , B_2 be two bounded subsets of G. The image of $B_1 \times B_2$ under product map denoted by $B_1 * B_2 = \{b_1 * b_2 : b_1 \in B_1, b_2 \in B_2\}$. Similarly, $B^{-1} = \{b^{-1} : b \in B\}$ is the image of B under the inverse map.

Example (2.2):

1) Let $(G, +)$ be an additive group and β be the collection of all finite subset, which it is finite bornology. To prove that (G, β) is a bornological group

1) $f : (G, \beta) \times (G, \beta) \to (G, \beta)$.

take $C, D \in (G, \beta)$. Then, C, D are finite sets in (G, β) (by definition of (G, β).

Thus, $f(C, D) = C + D$ is finite set . Thus, the image of C, D is bounded.

(2) Let $C \in (G, \beta)$, then $f^{-1}(C) = C^{-1} \subset G$ is finite set. Then, f^{-1} is bounded.

Example (2.3):

Afinite bornology on general linear group is a bornological group. That means $\mathrm{GL}(2,2) = (\begin{bmatrix} a & b \\ c & d \end{bmatrix}, +)$ with finite bornology is a bornological group the inverses element in this group $(\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix})$ To prove that, the product and inverse maps are bounds.

Let f : $(GL(2,2), \beta) \times (GL(2,2), \beta) \rightarrow (GL(2,2), \beta)$ is the product map.

Let M_1, M_2 are two bounded sets in $(GL(2,2), \beta)$, to prove that f $(M_1 \times M_2)$ is bounded.

$$f(M_1 \times M_2) = M_1 + M_2 = \{m_1 + m_2, m_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, m_2 = \begin{bmatrix} o & v \\ u & w \end{bmatrix} \in GL(2,2)\}$$

$$\in (GL(2,2), \beta) = \begin{bmatrix} a+o & b+v \\ c+u & d+w \end{bmatrix} : a, b, c, d, o, u, v, w \in R.$$

Thus, the product is bounded.

1) $f^{-1} : (GL(2,2), \beta) \rightarrow (GL\ (2,2), \beta)$.

Let $M \in (GL(2,2), \beta)$, $M = \{ m : m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\}$ which is finte set.

Then, $M^{-1} = -M = \{-m : -m = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}\} \in (GL(2,2), \beta)$.

So, f^{-1} is bounded. Then $(GL(2,2), \beta)$ is a bornological group.

Example (2.4):

Let $= (\{\begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}_{2\times 2} : e, f \in \mathbb{R}, e \neq 0\}, .)$, is a group with finite bornology β.

To prove G with finite bornology β is a bornological group (G, β).

i. $\psi : (G, \beta) \times (G, \beta) \longrightarrow (G, \beta)$ is bounded.

Let $D_1, D_2 \in (G, \beta)$ be two bounded subsets, we must prove that $\psi(D_1 \times D_2)$ is bounded.

$$\psi(D_1 \times D_2) = D_1 \cdot D_2 = \{d_1 \cdot d_2 : d_1 = \begin{bmatrix} e_1 & f_1 \\ 0 & 1 \end{bmatrix}, d_2 = \begin{bmatrix} e_2 & f_2 \\ 0 & 1 \end{bmatrix}, e_1, e_2, f_1, f_2 \in \mathbb{R}\}$$

$$=\{\begin{bmatrix} e_1 & f_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_2 & f_2 \\ 0 & 1 \end{bmatrix}\} = \{\begin{bmatrix} e_1 \cdot e_2 & e_1 f_2 + f_1 \\ 0 & 1 \end{bmatrix}\} \in (G, \beta)$$

ii. $\psi^{-1} : (G, \beta) \longrightarrow (G, \beta)$.

Academic Science Journal

Let $D \in (G,\beta), D = \{d: d = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix}\}$ which is finite set.

Then $\psi^{-1}(D) = D^{-1} = \{d^{-1}: d^{-1} = \begin{bmatrix} \frac{1}{e} & \frac{-f}{e} \\ 0 & 1 \end{bmatrix}\} \in (G,\beta)$. So, ψ^{-1} is bounded. Then (G,β) is bornological group.

The differences between bornology and topology

1. **Bornology:** Let X be a set, any collection of subsets of X is a bornology β on X if satisfyies three conditions

i. β covers X. We can covers any set by deferent ways

a. If $X \in \beta$, since the set X belong to the collection β, then X covers itself.

b. $\forall x \in X, \{x\} \in \beta$, then β cover X.

c. $X = \bigcup_{B\in\beta} B$, then β cover X.

ii. β stable under hereditary ,If $B \in \beta$ and $A \subseteq B$,then $A \in \beta$.

iii. β stable under finite union.

Topology: Let X be a set any collection of subsets of X is topology τ on X if satisfies three condition.

i. $X, \emptyset \in \tau$.

ii. τ stable under finite intersection :If $O_1, O_2 \in \tau$,then $O_1 \cap O_2 \in \tau$.

iii. τ stable under infinite union.

Some notes

i. Can we replace the first condition with $X \in \beta$ (since if the set belong to the collection, then it can cover it self) like topology($X \in \tau$)?

Answer: No, we cannot. because if we want to cover the infinite set X by the usual bornology, collection of all usual bounded set ,the usual bounded is the usual concept of bounded set with respct to this set, then $X \notin \beta$, because, the infinite set is not usual bounded set.

ii. Can replace the hereditary properties in bornology by finite intersection?

Answer: No, because the finite intersection give just the smaller set. But, the hereditary properties can include all the subsets of the bounded set B and that exactly what the bornology structure working on.

iii. Can replace the third conditionin bornology by infinite union instead of finite union**?**

Answer: No, the reader can understand the reason when they understand the practical application for bornology.

2. **Bornology:** The element of bornology is called bounded set.

Topology: The element of topology is called open set.

3. **Bornology:** When defined bornology on a set X, then the pair(X, β) is called bornological set.

Topology: When define topology on set X, then the pair(X, τ) is called topological space.

4. **Bornology:** In bornological set, they just the fundamental construction of bornology , For example the base, sub set... ealt

Topology: In topological space, the fundamental construction and properties are studied, For example, compact, complete, closed set, closure set, converge of sequences and separation axiom.

5. **Bornology:** in defining bornology on a vector space is called bornological space and can study the properties **Topology:** When defining topology on a vector space is called topological vector space.
6. **Bornology:** Any sub collection $\beta_\circ$ is a base if for every element of β is contain in an element of $\beta_\circ$.

Topology: Any sub collection $\tau_\circ$ is a base if for every element of $\tau_\circ$ is contain in an element of τ.

7. **Bornology:** The morphism between two bornological sets are called bounded map.

Topology: The morphism between two topological spaces are called continuous map.

8. **Bornbology:** There is no trivial bornology.

Topology: Trivial topology. $\tau = \{\emptyset, X\}$.

9. **Bornology:** We can define just discrete bornology on finite set.

Topology: We can define many topologies on finite set.

10. **Bornology:** A map $\boldsymbol{f}$ between two bornological sets are isomorphism if it is bijective and $\boldsymbol{f}, \boldsymbol{f}^{-1}$ are bounded**.**

Topology: A map f between two topological space are homeomorphism if it is bijective and $\boldsymbol{f}, \boldsymbol{f}^{-1}$ are continuous.

11. **Bornology:** Solving the problem of bounded in sets

Topology: Solving the continuity of shapes.

3. Ideal Bornological Set.

In this section, a new structure, which is called ideal bornological set are constructed, to solve the problem of bounds in ideal way. Furthermore, the fundamental construction for this structure are considered. First of all, any collection or family of subsets of X is called ideal on X if satisfying two conditions. First, having hereditary properties. Second, finite addition property. In other words, a family on a set X is an ideal if it has hereditary property and is stable under finite union.

Definition (3.1):

Let X be a set and $\boldsymbol{\beta}$ a bornology on X. An ideal bornology on a set X, which is denoted by $\boldsymbol{\beta_I}$ is a family of subsets of X satisfying the following two properties:

1. If $\bar{B} \in \beta_I$, $C \subseteq \bar{B}$, then $C \in \beta_I$;
2. If $\bar{B}, \bar{\bar{B}} \in \beta_I$, then $\bar{B} \cup \bar{\bar{B}} \in \beta_I$.

If (X, β) is a bornological set and β_I is an ideal bornology on X, then the symbol (X, β, β_I) are said to be an ideal bornological set. Now, let discuss some types of ideal bornological set.

- $\beta_{I\emptyset} = \{\emptyset\}$ is the trivial ideal bornological set.
- $\beta_{I\rho} = \rho\,(X)$ is the principal ideal equal to P(X). Which is the collection of all subset of X.

3- β_{If}= { $A \subseteq$ X: A is a finite set, where X is infinite} it is the ideal bornological set of all finite subsets of X. Which it is called finite ideal bornological set.

4-β_{Ic}= { A ⊆ X : A is countable set} is the ideal bornological set of all countable subsets of X, which it is called countable ideal bornological set.

Example (3.2):

Let $\boldsymbol{X} = \{\mathbf{d}, \mathbf{e}, \mathbf{f}\}$ with discrete bornology, which it is the collection of all subsets of X. Then, there is more than ideal bornological set can be define as follows:$\boldsymbol{\beta_{I1}} = \{\{\mathbf{d}\}, \{\mathbf{e}\}, \{\mathbf{d}, \mathbf{e}\}\}$; $\boldsymbol{\beta_{I2}} = \{\{\mathbf{d}\}, \{\mathbf{f}\}, \{\mathbf{d}, \mathbf{f}\}\}$; $\boldsymbol{\beta_{I3}} = \{\{\mathbf{e}\}, \{\mathbf{f}\}, \{\mathbf{e}, \mathbf{f}\}\}$.

Remark (3.3):

Every ideal bornological set is sub collection of bornological set.

Definition (3.4):

Let (A, β, β_I), $(\bar{A}, \bar{\beta}, \bar{\beta}_I)$ be ideals bornological sets, a map $f: (A, \beta, \beta_I) \to (\bar{A}, \bar{\beta}, \bar{\beta}_I)$ is called an ideal bounded map if the image for every bounded sub set in (A, β, β_I) is bounded set in $(\bar{A}, \bar{\beta}, \bar{\beta}_I)$.

Proposition (3.5):

Let $(\boldsymbol{A}, \boldsymbol{\beta}, \boldsymbol{\beta_I})$, $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta_I}})$ and $(\bar{\bar{\boldsymbol{A}}}, \bar{\bar{\boldsymbol{\beta}}}, \overline{\overline{\boldsymbol{\beta_I}}})$ be ideal bornological sets and $\boldsymbol{f}: (\boldsymbol{A}, \boldsymbol{\beta}, \boldsymbol{\beta_I}) \to (\bar{\boldsymbol{A}}, \bar{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta_I}})$, $\boldsymbol{g}: (\bar{\boldsymbol{A}}, \bar{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta_I}}) \to (\bar{\bar{\boldsymbol{A}}}, \bar{\bar{\boldsymbol{\beta}}}, \overline{\overline{\boldsymbol{\beta_I}}})$ be any ideal bounded maps, then their composition $\boldsymbol{g} \circ \boldsymbol{f}: (\boldsymbol{A}, \boldsymbol{\beta}, \boldsymbol{\beta_I}) \to (\bar{\bar{\boldsymbol{A}}}, \bar{\bar{\boldsymbol{\beta}}}, \overline{\overline{\boldsymbol{\beta_I}}})$ is an ideal bounded map.

Proof: Suppose $B \in \beta_I$, since f is an ideal bounded map, then $f(B)$ is bounded set in $\bar{\beta}_I$. Also, since g is an ideal bounded map. Then, $g(f(B)) \in \bar{\bar{\beta}}_I$. So, the composition $g \circ f$ is an ideal bounded map.

Proposition (3.6):

Let $\bar{\boldsymbol{\beta}}_{\boldsymbol{I}}, \bar{\bar{\boldsymbol{\beta}}}_{\boldsymbol{I}}$ be two ideal bornological sets on $\boldsymbol{X}$, then $(\boldsymbol{X}, \bar{\boldsymbol{\beta}}_{\boldsymbol{I}} \cap \bar{\bar{\boldsymbol{\beta}}}_{\boldsymbol{I}})$ is an ideal bornological set.

Proof:1-Since $\bar{\beta}_I$ and $\bar{\bar{\beta}}_I$ are ideal bornological sets, that means, they are stable under hereditary. Let $B \in \bar{\beta}_I \cap \bar{\bar{\beta}}_I$, then $B \in \bar{\beta}_I$ and $B \in \bar{\bar{\beta}}_I$.Since $\bar{\beta}_I, \bar{\bar{\beta}}_I$ are ideal bornological sets. Then, there exist $M \subseteq B$, where $B \in \bar{\beta}_I$, $\bar{\bar{\beta}}_I$.Thus, $M \in \bar{\beta}_I$ and $M \in \bar{\bar{\beta}}_I$. So $M \in \bar{\beta}_I \cap \bar{\bar{\beta}}_I$. Then $\bar{\beta}_I \cap \bar{\bar{\beta}}_I$ have hereditary properties.

2- Let $A, B \in \bar{\beta}_I \cap \bar{\bar{\beta}}_I$ and $A, B \in \bar{\beta}_I$ and $A, B \in \bar{\bar{\beta}}_I$. Since, $\bar{\beta}_I, \bar{\bar{\beta}}_I$ be ideal bornological sets. Then, $A \cup B \in \bar{\beta}_I$ and $A \cup B \in \bar{\bar{\beta}}_I$. So, $A \cup B \in \bar{\beta}_I \cap \bar{\bar{\beta}}_I$. Thus $(X, \bar{\beta}_I \cap \bar{\bar{\beta}}_I)$ is an ideal bornological set.But the union of two ideal bornological sets is not ideal bornology.

Example (3.7):

Let $X = \{1,2,3\}$ with discrete bornology on X, and $\beta_{I1} = \{\{1\},\{2\},\{1,2\}\}$, $\beta_{I2} = \{\{1\},\{3\},\{1,3\}\}$ are ideal bornological sets, but $\beta_{I1} \cup \beta_{I2} = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\}\}$ are not.

Definition (3.8):

A map between two ideal bornological sets (X, β, β_I), $(\bar{X}, \bar{\beta}, \bar{\beta}_I)$ is called an ideal bornological isomorphism if it is bijective map and f, f^{-1} are ideal bounded maps.

Definition (3.9):

Let (X, β, β_I) be an ideal bornological set and $Y \subseteq X$ .Then the collection $\beta_{IY} = \{V \cap Y : V \in \beta_I\}$ is an ideal bornology on Y. Such that (Y, β_Y, β_{IY}) is called sub ideal bornogical set of (X, β, β_I).

Proposition (3.10):

If (X, β, β_I) be an ideal bornological set and $Y \subseteq X$, then $\beta_{IY} = \{V \cap Y : V \in \beta_I\}$ is a sub ideal bornology on Y.

Proof:

- Take $B \in \beta_{IY}$, i.e., $B = V \cap Y$, where $V \in \beta_I$. Let $C \subseteq Y, C \subseteq B$. To prove $C \in \beta_{IY}$, we must prove that $C = U \cap Y$, where $U \in \beta_I$. $C \subseteq V \cap Y$, $C \subseteq V$ and $C \subseteq Y$. Since $V \in \beta_I$, then $C \in \beta_I$.Take $C = U$, such that $C = U \cap Y$ and $U \in \beta_I$. Then, $C \in \beta_{IY}$ and β_{IY} satisfy hereditary properties.
- Let $\{B_i\} \in \beta_{IY}$. To prove that $\bigcup_{i=1}^{n}(B_i) \in \beta_{IY}$. Since $\forall$ i=1,…..,n $\exists$ $V_i \in \beta_I$, s.t. $B_i = V_i \cap Y$, then $\bigcup_{i=1}^{n}(B_i) = \bigcup_{i=1}^{n}(V_i \cap Y) = (\bigcup_{i=1}^{n} V_i) \cap Y = V \cap Y$. When $(\bigcup_{i=1}^{n} V_i) \in \beta_I$ (Since β_I is stable under finite union). Then $V \cap Y \in \beta_{IY}$, i.e. $\bigcup_{i=1}^{n}(B_i) \in \beta_{IY}$. It is clear that every sub ideal bornology β_{IY} is a sub collection of ideal bornology β_I , i.e. $\beta_{IY} \subseteq \beta_I$.

Academic Science Journal

Definition (3.11):

If (X, β, β_I) is an ideal bornological set, then $\beta_{\circ I}$ is an **ideal base** when every element of the ideal bornology is contained in an element of the ideal base.

Example (3.12):

Let $X = \{1,2,3\}$, it is clear that $\beta_I = \{\{1\},\{2\},\{1,2\}\}$ is an ideal bornology, and the ideal base for β_I is $\beta_{\circ I} = \{\{1, 2\}\}$.

Proposition (3.13):

If $(\boldsymbol{X}, \boldsymbol{\beta}, \boldsymbol{\beta_I})$ be an ideal bornological set and $\boldsymbol{\beta_{\circ I}}$ is the ideal base of $\boldsymbol{\beta_I}$, then the ideal base is stable under finite union.

Proof: Let $B_{\circ 1}, B_{\circ 2} \in \beta_{\circ I}$ .Then there exist $B_1, B_2 \in \beta_I$, such that $B_1 \subseteq B_{\circ 1}, B_2 \subseteq B_{\circ 2}$ and $B_1 \cup B_2 \subseteq B_{\circ 1} \cup B_{\circ 2}$ .From the definition of the ideal base ($\forall B_\circ \in \beta_{\circ I}, \exists B \in \beta_I$, such that $B \subseteq B_\circ$). Since β_I, is stable under finite union, thus $B_1 \cup B_2 \in \beta_I$. Then, $B_{\circ 1} \cup B_{\circ 2} \in \beta_{\circ I}$.

The following proposition shows that, there is a bornology generated by given ideal base.

Proposition (3.14):

There is an ideal bornology β_I generated from the ideal base $\beta_{\circ I}$, such that $\beta_I = \{B \subseteq X: \exists B_\circ \in \beta_{\circ I}, B \subseteq B_\circ\}$.

Proof: To show that β_I is an ideal bornology and since any collection stable under hereditary and finite union it will be an ideal bornology. We must prove the following two conditions:

- Let $B \in \beta_I$, and $C \subseteq X$, $C \subseteq B$, since $B \subseteq B_\circ$, $B_\circ \in \beta_{\circ I}$. Then $C \subseteq B_\circ$. So, $C \in \beta_I$. Since C contained in an element of $\beta_{\circ I}$ (by definition of β_I). Thus, $C \in \beta_I$.
- Now if $B_1, B_2 \in \beta_I$. Then $\exists B_\circ, \bar{B}_\circ \in \beta_{\circ I}$. Such that, $B_1 \subseteq B_\circ$ and $B_2 \subseteq \bar{B}_\circ$. Then $B_1 \cup B_2 \subseteq B_\circ \cup \bar{B}_\circ \subseteq B_\circ$, for some $B_\circ \in \beta_{\circ I}$ (by definition of β_I). Then β_I is an ideal bornology generated by the ideal base.

Definition (3.15):

Let $(X, \beta_1, \beta_{I1}), (Y, \beta_2, \beta_{I2})$ be two ideal bornological sets, that β_{I1} is finer than β_{I2} and β_{I2} is coarser than β_{I1} if $\beta_{1I} \subseteq \beta_{2I}$.

Example (3.16):

Let X be an infinite set and the finite ideal bornology β_{If} is a family of all finite set of X and β_{IP} is a family of all subset of X. Then β_{If} is finer than principle ideal bornology β_{IP} on X.

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