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New Fixed Point Theorem for Expansive Maps on a Two – Metric Space

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Abstract

In this paper new theorems a bout fixed point are given for applications in 2- metric space. The common fixed points of two operators exhibit a shared periodicity in their occurrences. The theorem demonstrates that, certain conditions, there exists a specific pattern or sequence of operators where the fixed points exhibit a common periodicity.

Keyword: Two – Metric Space, fixed point, applications in 2- metric space, theorems in Two – Metric Space

نظريات جديدة للنقطة الثابتة لتطبيقات معرفة على الفضاء المتري الثنائي

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الخلاصة

هدفت الدراسة لعرض نظريات جديدة ، للتطبيقات نقطة ثابتة في 2- الفضاء مترية. مثل النقاط الثابتة الدورية المشتركة بين عمليتين ، ، او سلسلة من العمليات تحت شروط نظريات موضوع البحث.

الكلمات مفتاحية: الفضاء المتري الثنائي ، النقطة الثابتة ، تطبيقات في الفضاء المتري الثنائي، نظريات في الفضاء المتري الثنائي

Introduction

A metric space (X, d) is familiar for B.Sc. Students in mathematic where the metric function d is not unique and if (X, d_1) and (X, d_2) are matrices spaces with X as finite dimensional than (X, d_2) and (X, d_2) are equivalent. In initial value problem we use the complete metric space (X, d) where X consists of the solution and d as contraction to get unique fixed point that is to get unique solution.

There are some generalization for a metric space and the fixed point , such as D-metric space, cone metric space ,complex value metric space , G.metric space and 2-metric space[4,19,525,26,27,28,29,30], 2-metric space and ordinary metric are not topologically equivalent. Therefor the relationship between the obtained results is not easy in 2-metric space and metric space. Also the themes of fixed point between there the kinds of metric spaces are not related easily in general [4, 19, 27].

Extensive research has been carried out in the field. Duo 1963 and Gahler were the first to work in this space. A number of applications such as game theory, medicine, economic etc. The fixed point theorem in mathematics can be considered as a consequence for studing functions or operators on a spaces. A point x_0 is called fixed point for a function $f(x)$ if $f(x_0) = x_0$. Remember that functions can have no or one or finite or infinite a fixed point [1]. Such that $f(x) = x^2 + 1$, $f(x) = \sin x$, $f(x)$, x^2 , and $f(x) = x$ respectively, Banach and Brauer are the pioneers of this field. One can use fixed point theorem for proving the uniqueness of the set. of a given initial value problem under certain conditions. That it is used frequently in many fields such as dynamical system, physics, energy, etc. [2, 3]

Metric Space

Definition 1:

2- Metric space [7]: Suppose X is non empty set and suppose $d: X * X \to R$ satisfies the following conditions :

1. For a pair of points $x, y \in X$ there exists a point

$z \in X$ such that $d(x, y, z) \neq 0$.

2. For each $x, y, z \in X$ then $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$
3. For all $x, y, z, t \in X$ then the inequality
   $d(x,y,z) \leq d(x,y,t) + d(x,y,t) + d(y,z,t)$ holds
4. For all $x, y, z \in X$ then $d(x,y,z) = 0$ iff
   $x = y = z$

Definition 2[1]**:**

Convergence in 2 – metric space: Suppose (x_n) is a sequences in a 2- metric space (X, d) then we say that (x_n) converges to $x \in X$ if satisfies the following condition:

For all $a \in X$ then $\lim_{n\to\infty} d(x_n, x, a) = 0$.

Definition 3[6]**:**

Cauchy sequence in 2 – metric space (x_n) is cauchy sequence in 2 – metric space if satisfies the following conditions:

for all: $a \in X, n, m \in N$ then $\lim_{n,m\to\infty} d(x_n, x_m, a) = 0$

Theorems for fixed points in 2 – metric space

Theorem 1 [21]:

Suppose X is a complete 2 – metric space and suppose $f: X \to X$ satisfy for all:

$x, y, a \in X$,

$$d(f(x), f(y), a) \leq \alpha[d(x, f(x), a) + d(y, f(y), a)] + \beta d(x, y, a) + \gamma \max\{d(x, f(y), a), d(y, f(x), a\}.$$

s.t. $\alpha, \beta, \gamma \geq 0$, $2\alpha + \beta + 2\gamma < 1$ then the function f has a fixed point z and $\lim_{n\to\infty} f^n(x_0) = z$, $\forall\, x_0 \in X$.

Theorem 2 [12]:

Suppose X is complete 2- me*tric space and suppose* f_1 *and* f_2 *are two mappings in* X *then for all* $x, y, a \in X$ *:*

$d(f_1(x), f_2(y), a) \le a_1 d(x, f_1(x), a) + a_2 d(y, f_2(y), a) + a_3 d(x, f_2(y), a) + a_4 d(y, f_1(x), a) + a_5 d(x, y, a)$

such that $a_{1,} a_2, a_3, a_4$ and a_5, are non-negative numbers satisfy $\sum_{i=1}^{5} a_i < 1$ and $(a_1 - a_2), (a_3 - a_4) \ge 0$ then f_1 and f_2 have a common fixed point .

Theorem 3 [21]:

Suppose (X, d) is complete 2-metric space and suppose f is continuous mapping from X s.t. $f: X \to X$ and satisfy the following conditions

$d^2(fx, fy, a) \le \alpha d(x, fx, a) d(y, fy, a) + \beta d(x, fx, a) d(y, fx, a) + \gamma d(y, fy, a) d(y, fx, a) + \delta d(x, fy, a) d(y, fx, a),$

for all $x, y, a \in X$ and $\alpha, \beta, \gamma, \delta \ge 0$ and $\max(\alpha, \delta) < 1$ then f has fixed point in X.

Theorem 4[21]:

Suppose (X, d) is complete 2-metric space and suppose f, T are two continuous identical mappings from X and satisfy the following conditions:

1. $fT = Tf; f(X) \subset T(X)$
2. $d^2(fx, fy, a) \le \alpha d(Tx, fx, a) d(Ty, fy, a) + \beta d(Tx, fx, a) d(Ty, fx, a) + \gamma d(Ty, fy, a) d(Ty, fx, a) + \delta d(Tx, fy, a) d(Ty, fx, a).$

For all $x, y, a \in X$ and $\alpha, \beta, \gamma, \delta \ge 0$ with $\max(\alpha, \delta) < 1$ then f, T has a common fixed point in X.

Theorem 5[21]:

Suppose (X, d) complete 2-metric space and suppose E, T, F continuous functional from X and satisfy the following conditions:

1. $ET = TE; FT = TF; E(X) \subset T(X); F(X) \subset T(X)$
2. $d^2(Ex, Fy, a) \le \alpha d(Tx, Ex, a) d(Ty, Fy, a) + \beta d(Tx, Ex, a) d(Ty, Ex, a) ET + \gamma d(Ty, Fy, a) d(Ty, Ex, a) + \delta d(Tx, Fy, a) d(Ty, Ex, a)$

For all $x, y, a \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ with $\max(\alpha, \delta) < 1$ then E, T, F have a commn fixed point in X.

Convergence theorems of sequences of mappings in a 2-metric space and their fixed points:

Theorem 1 [18]

Suppose X is complete 2-metric space and d is continuous and suppose $\{f_n\}$ sequence of mappings. Defined by $f_n: X \to x$ and satisfy the following conditions:

$$d(f(x), f(y), a) \leq \alpha[d(x, f(x), a) + d(y, f(y), a)] + \beta d(x, y, a) + \gamma \max\{d(x, f(y), a), d(y, f(x), a)\}$$

For $n \in N$ and the numbers $\alpha, \beta, \gamma \in \mathbb{R}^+$ also $\{f_n\}$ converges point wise to f , then f has a fixed point z and such that $z_n \to z$ then z_n is the fixed point of f_n .

Theorem 2 [21]

Suppose X complete 2-metric space and suppose $\{f_n\}$ sequence of functions $f_n: X \to X$ has fixed point z_n with the following conditions:

$$d(f(x), f(y), a) \leq \alpha[d(x, f(x), a) + d(y, f(y), a)] + \beta d(x, y, a) + \gamma \max\{d(x, f(y), a), d(y, f(x), a)\}$$

has fixed points z and such that $f_n \to f$ there regularly $z_n \to z$.

Main Results

Through the following five theorems we prove the existance of fixed points of operators defend on 2- metric spaces and also a common fixed point for operators under certain conditions as shown in these theorems

Theorem 1

Suppose X is complete 2-metric space and suppose $T: X \to X$ satisfies:

$$d(Tx, Ty, a) \geq \sqrt{d(x, Tx, a) d(y, Ty, a)}$$

For all, $y, a \in X$, then T has periodic point and T continuous functional.

Proof:

Define a sequence (x_n) from X such that $x_n = T^n x_0$

for some value x_0 suppose $x_n \neq x_m$ whatever m, n then:

$$d(x_n, x_{n+1}, a) = d(Tx_{n-1}, Tx_n, a) \geq \sqrt{d(x_{n-1}, Tx_{n-1}, a) d(x_n, Tx_n, a)}$$

$$= \sqrt{d(x_{n-1}, x_n, a) d(x_n, x_{n+1}, a)}$$

or $d(x_n, x_{n+1}, a) \geq d(x_{n-1}, x_n, a)$.

Similarly: $d(x_{n-1}, x_n, a) \geq d(x_{n-2}, x_{n-1}, a)$ and so we find:

$$d(x_n, x_{n+1}, a) \geq d(x_{n-1}, x_n, a) \geq \cdots \geq d(x_0, x_1, a)$$

and so for all positive numbers r, n we find:

$$d(x_n, x_{n+r}, a) \geq \sqrt{d(x_{n-1}, x_n, a) d(x_{n+r-1}, x_{n+r}, a)}$$
$$\geq d(x_0, x_1, a) > 0$$

That is $d(x_n, x_{n+r}, a) \geq 0$

This shows that the sequence (x_n) is not convergent and this is a contradiction so $x_n \neq x_m$is not for all values of m, n so we have $x_n = x_{n+r}$ for some values of $n \geq 0, r \geq 1$ then $x_n = T^r x_n$.

Now let's say n is the smallest non-negative integer such that $x_n = x_{n+r}$ so we'll choose the smallest number r, let's say $n \geq 1$ then the sequence $x_0, x_1, \dots \dots, x_{n+r-1}$ consists of different points.

Now we have:

$$0 = d(x_n, x_{n+r}, a) \geq \sqrt{d(x_n, x_{n-1}, a) d(x_{n+r}, x_{n+r-1}, a)}$$
$$\geq d(x_1, x_0, a) \geq 0$$

It is axiomatic that $n = 0$ so $x_0$is a periodic point of T.

Let's say $T_x = T_y$ for some $x \neq y$ thus:

$$0 = d(Tx, Ty, a) \geq \sqrt{d(x, Tx, a) d(y, Ty, a)}$$

This gives that $x = T_x$ or $y = T_y$ and since y is periodic we have $T^m y = y$ for some values of > 0 , now for $x = T_x$ and $y = T^m y = T^{m-1} T_y = T^{m-1} T_x = x$ that is $x = y$ and this is a contradiction.

Similarly, $T_y = y$ leads to contradiction, which means that T is a discrete implementation.

Theorem 2:

Suppose $S, T: X \to X$are continuous mappings on a complete 2-metric space X in which:

$$d(Sx, Ty, a) < \sqrt{d(x, Sx, a) d(y, Ty, a)}$$

So for all $x, y, a \in X$ and $x \neq y \neq a$ then S and T have a common fixed point and X is a singular space.

Proof:

Define the sequence (x_n) of X so that $x_{2n+1} = Sx_{2n}, x_{2n} = Tx_{2n-1}$ so X is a compact space so (x_n) has partial chains x_{nk} such that $x_{nk} \to x$ in X for some x of X.

Now from the continuation S, T we have:

$$x = \lim_{k\to\infty} x_{n_{2k}} = \lim_{k\to\infty} Tx_{n_{2k-1}} = T\left(\lim_{k\to\infty} x_{n_{2k-1}}\right) = Tx \text{ or } x = Tx$$

Similarly $x = S_x$.

We will now prove that X is singular so let's put $y \neq x$ of X then for each a from X we have :

$$d(x, Ty, a) = d(Sx, Ty, a) < \sqrt{d(x, Sx, a)d(y, Ty, a)} = 0$$

Or $d(x, T_y, a) < 0$ for each a of X which is not possible therefore $x = y$

Theorem 3:

Assuming X is a 2-metric space and assuming $T: X \to X$, such that:

$$d(Tx, Ty, a) \geq \frac{d^2(Tx, x, a) + d^2(Ty, y, a)}{d(Tx, x, a) + d(Ty, y, a)}$$

and that for all $a \in X$ and $d(T_x, x, a) + d(T_y, y, a) \neq 0$ then every point x of X is a fixed point of T.

Proof:

Assuming x is some point of X then for each $\in X$:

$$0 = d(Tx, x, a) \geq \frac{d^2(Tx, x, a) + d^2(Tx, x, a)}{d(Tx, x, a) + d(Tx, x, a)} = d(Tx, x, a)$$

That is, $d(Tx, x, a) = 0$ for each $a \in X$ or $Tx = x$.

Theorem 4:

Suppose S_n ,$T_n: X \to X$ are a sequences of mappings such that:

$$d(S_m x, T_n y, a) \leq h\sqrt{d(S_m x, x, a)d(T_n y, y, a)}$$

This is for all $x, y, a \in X$ and $0 < h < 1$ then for all values of m, n for which S_n and T_n have a common fixed point.

Proof: Let's define the sequence (x_n) of X, so that $x_{2n+1} = S_{n+1} x_{2n}, x_{2n} = T_n x_{2n-1}$.

If $x_{2n} \neq x_{2n+1}$ then for each a of X we have:

$$d(x_{2n+1}, x_{2n}, a) = d(S_{n+1}x_{2n}, T_n x_{2n-1}, a)$$
$$\leq h\sqrt{d(S_{n+1}x_{2n}, x_{2n}, a)d(T_n x_{2n-1}, x_{2n-1}, a)}$$
$$= h\sqrt{d(x_{2n+1}, x_{2n}, a)d(x_{2n}, x_{2n-1}, a)}$$

That is: $d(x_{2n+1}, x_{2n}, a) \leq \rho d(x_{2n}, x_{2n-1}, a); \;\; \rho = h^2 < 1$

Similarly: $d(x_{2n}, x_{2n-1}, a) \leq \rho d(x_{2n-1}, x_{2n-2}, a); \;\; \rho = h^2 < 1$

In general : $d(x_{n+1}, x_n, a) \leq \rho d(x_n, x_{n-1}, a)$

or : $d(x_{n+1}, x_n, a) \leq \rho^n d(x_1, x_0, a)$

Then $n \to \infty$ find $\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0$ so that $\rho < 1$.

It is easy to prove that (x_n) is the Cauchy sequence in X and that $(x_n) \to x; x \in X$.

Now let's take $d(x_{2n+1}, T_m x, a) = d(S_{n+1}x_{2n}, T_m x, a)$

$$\leq h\sqrt{d(S_{n+1}x_{2n}, x_{2n}, a)d(T_m x, x, a)}$$
$$= \sqrt{d(x_{2n+1}, x_{2n}, a)d(T_m x, x, a)}$$

When $n \to \infty$ we find $d(x, T_m x, a)$ or $x \in T_m x$ for all values of m and similarly $x \in S_m x$ this means that S_n, T_n has a common fixed point.

Theorem 5:

Assuming that X is a 2-metric space with two distance functions e, d if X satisfy the conditions:

1. X is a complete metric space with the distance e.
2. $e(x, y, a) \leq d(x, y, a)$
3. $S, T_n: X \to X$ and S continuing and checking $(Sx, T_n y, a) \leq \sqrt{d(Sx, x, a)d(T_n y, y, a)}$

So for all x, y, a $\in$ X and 0<h<1, then for all,S and T_n has a common fixed point.

Proof: Let us know the sequence (x_n) from X, so that $x_{2n+1} = Sx_{2n}, x_{2n} = T_n x_{2n-1}$ then according to the theorem (4), (x_n) will be a Cauchy sequence with a distance function d and only (2) in (x_n) is a Cauchy sequence with a space function e as well, so $(x_n) \to x; x \in X$.

From the continuum of S we have $x = S_x$.

Put $d(x, T_n x, a) = d(Sx, T_n x, a) \leq h\sqrt{d(Sx, x, a)d(T_n x, x, a)}$

$$= d(x, T_n x, a) = 0$$

That is for every a ∈ X or $x = T_n x$ for every n, so S, T_n have a common fixed point.

Conclusion

From the proved five theorems 2.4.1……. 2.4.5 we conclude when the 2-metric space is complete then the operator under certain condition is continuous and each point is periodics . If two operators are continuous and satisfy a given condition then they have common fixed point. Similarly for a sequencesal operators.

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