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Using simulation to find the best reliability of the Generalized Exponential Distribution

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Abstract

The generalized exponential distribution is one of the most widely used distributions in the field of geometric reliability. Also, it can be used as an alternative to the Gamma or Weibull distribution in many cases. The aim of this research is to find the best reliability function for the shape parameter (θ) of the Generalized Exponential Distribution using different methods (classical and Bayes) by using two non–information prior (Jeffery and Modify Jeffery) with different loss functions (squared error and Precautionary). The Reliability was based on the Monte Carlo study to use the simulation of the Reliability study on the performance of these estimations by using the mean square error.

Keywords: Generalized Exponential Distribution, Classical methods, Bayes methods, Mean Square Error, Jeffery prior Distribution, Modify Jeffery prior Distribution, Loss Functions.

استخدام المحاكاة لايجاد أفضل معمولية للتوزيع الاسي المعمم

هدى عامر عبد الامير

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الخلاصة

يعد التوزيع الأسي المعمم أحد التوزيعات الأكثر استخدامًا في مجال المعمولية الهندسية. كما يمكن استخدامه كبديل لتوزيع كاما أو ويبل في كثير من الحالات. الهدف من هذا البحث ايجاد افضل مقدر للمعلمة الشكل θ للتوزيع الاسي المعمم بأستخدام طرائق تقدير مختلفة (الكلاسيكية وبيزية) وباستخدام اثنين من التوزيعات الاولية (جيفري و جيفري المعدل) مع دوال الخسارة المختلفة (مربع الخطأ والوقائية). وتم الاعتماد على دراسة مونت كارلو لاستخدام المحاكاة لدراسة ثباتية أداء هذه التقديرات باستخدام متوسط مربع الخطأ.

الكلمات المفتاحية : التوزيع الأسي المعمم، الطرق الكلاسيكية، طرق بيزية، متوسط مربع الخطأ، توزيع جيفري الاولي ، توزيع جيفري المعدل الاولي ، دوال الخسارة.

Introduction

The Generalized Exponential Distribution of two parameters is used to analyses the data of lifetime for the produced units until the failure happens, for example, the various equipment and vehicles. In 1999, this distribution was first introduced by Cupta and Kundu, [1]. The function of the Probability Density Function (PDF), for $x > 0$, is

$$f(x;\theta,\lambda) = \theta\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\theta-1} \quad \theta,\lambda > 0 \qquad (1)$$

where θ is the shape parameter and λ is the scale parameter. Also, the Cumulative Distribution Function (CDF) is given as:

$$F(x;\theta,\lambda) = \left(1 - e^{-\lambda x}\right)^{\theta} \quad \theta,\lambda > 0 \qquad (2)$$

The reliability function, $R(t)$ is given as follows:

$$R(x) = 1 - \left(1 - e^{-\lambda x}\right)^{\theta} \quad x > 0, \theta > 0, \lambda > 0 \qquad (3)$$

The Generalized Exponential Distribution has been studied by several researchers. For example, In 2006, two parameters of the general exponential distribution were estimated by using the two methods, MLE and ME. The comparison of them is performed by applying the simulation when all of the parameters are unknown depending on the bias amount and the mean squared error as a basis for comparison, [2]. In 2007, acceptance sampling plans under the name Economic Reliability Test Plans (ERTP) is designed by assuming that the lifetime of the test follows the general exponential distribution of the three parameters (α , λ, μ) for a specified time (T), [3]. In 2008, the Bayes estimates for two parameters of the general exponential distribution were studied as unknown by the assumption that the previous distribution is the Gamma Distribution, [4]. Furthermore, the approximation estimations were calculated by using Lindley's idea. The results of the Bayes estimates and the estimates of the maximum likelihood were compared by using simulation. The researchers concluded that the Bayes estimates were extremely better than the estimates of the maximum likelihood when the size of the sample is increased, [4]. In 2010, two groups of acceptance sampling plans are designed when the truncation time (the time taken until failure) follows the general exponential distribution GED $(t; \propto, \lambda)$. The first group is called the original group which assumes that the batch is rejected if there are c units of failure greater than the specified one registered in any set g of the two groups, [5]. In 2018, the maximum likelihood (ML) methods along with the method of moments are considered to estimate the parameters. Deriving the method of moments led to obtaining sufficient conditions for the existence of a unique solution of the parameters. Finally, a common numerical example is illustrated to investigate the application of both methods of estimation. The results obtained there were compared with four similar well-known three-parameter distributions, [6]. In 2022, type I generalized progressive hybrid censoring data from generalized exponential distribution is used [7]. Moreover, the maximum likelihood and Bayesian estimators of the distribution's parameters as well as the reliability and hazard functions are approximately calculated [7]. In 2023, the researchers used the Generalized Exponential Distribution (GED) to investigate a conservative imputation technique to deal with interval-censored data [8]. The comparison of the bias, standard error (SE) and the root mean square error (RMSE) of the maximum likelihood estimation (MLE) are done. The results showed that the proposed imputation method provides is more accurate than the classical method [8].

The Classical Methods of Maximum Likelihood Estimator (MLE) and Percentiles Estimator (PERCE)

The second group, called the improved group, assumes that the batch is accepted if the failed units for each group are less than the number of the specified acceptance; therefore, the sample is reduced.

Let $x_1, x_2, \cdots, x_n$ be random samples of size n that are taken from the general exponential distribution (GED), the maximum likelihood estimation is given as, [9],

$$\hat{\theta}_{MLE} = -\frac{n}{\sum_{i=1}^{n} ln(1-e^{-\lambda x_i})} \quad (4)$$

By noting that the Reliability Function $R(t)$ is defined as the probability of the system will operate at a period X, [10]. Since the maximum likelihood estimator is one-to-one mapping and an invariant, the maximum likelihood of the reliability function is denoted by

$$\hat{R}(x)_{MLE} = 1 - \left(1 - e^{-\lambda x}\right)^{\hat{\theta}_{MLE}} \tag{5}$$

Now, for the general exponential distribution, the percentiles point estimators can be used to estimate the parameter θ when λ is known, see [9].

$$\hat{\theta}_{PCE} = \frac{\sum_{i=1}^{n}(\ln(p_i)\ \ln(1-e^{-\lambda x_i})}{\sum_{i=1}^{n}(\ln(1-e^{-\lambda x_i}))^2} \tag{6}$$

Since equation (5) depends on the estimator (p_i), then

$$p_i = \frac{i-3/8}{n+1/4} \tag{7}$$

$$p_i = \frac{\left(i-\frac{1}{2}\right)}{n}$$

The approximate Percentiles Estimator of $R(x)$ denoted by $\hat{R}(x)_{PCE}$ is given by

$$\hat{R}(x)_{PCE} \approx 1 - \left(1 - e^{-\lambda x}\right)^{\hat{\theta}_{PCE}} \tag{8}$$

Bayes Estimators

In the Bayesian estimation procedure, we consider the parameter (or parameters) as a random variable (random variables) that has a prior distribution determined from the experience and the past data.

That is, we use the priors $g(\theta)$ or $g(\theta_1), g(\theta_2) \cdots g(\theta_n)$ to find the posterior distribution of the parameters that are given by the observations of the samples

$f(\theta/x)$. Then, according to the obtained posterior and different types of loss functions, we can obtain the Bayesian estimator of θ, by noting that the Bayesian estimator of θ minimizes the expected loss function.

We consider having the prior distributions Jeffery and Modify Jeffery to obtain two different types of posterior distributions. Moreover, by using the obtained posterior distributions with the loss functions (the squared error loss function and precautionary loss function) we can find the Bayesian Reliability Estimators [6].

1. Prior and Posterior Density Function

We must define a prior distribution for the parameter to perform Bayesian estimation. That is, we take Modified Jeffery prior and Jeffery prior information into consideration [2]. The posterior density function of θ written as denoted by $\prod(\theta|\underline{x})$. Also, for the given random sample X, we combine the specified prior with the likelihood, which is given by Equation (5), as below, [11].

$$\pi(\theta|\underline{x}) = \frac{\prod_{i=1}^{n} f(x_i;\theta).g(\theta)}{\int \prod_{i=1}^{n} f(x_i;\theta).g(\theta)d(\theta)} \tag{9}$$

Note that, the prior and the corresponding posterior density function are summarized by solving Equation (9). Then,

$$\pi(\theta|\underline{x}) = \frac{\prod_{i=1}^{n} f(x_i;\theta).g(\theta)}{\int \prod_{i=1}^{n} f(x_i;\theta).g(\theta)}$$

$$= \frac{\theta^n \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(1-e^{-\lambda x_i})^{\theta-1} \frac{1}{\theta}}{\int_0^\infty \theta^n \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(1-e^{-\lambda x_i})^{\theta-1} \frac{1}{\theta} \quad d\theta}$$

$$\frac{\theta^{n-1} e^{-\theta \sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}}}{\int_0^\infty \theta^{n-1} e^{-\theta \sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}} d\theta}$$

$$\pi(\theta|\underline{x})_j = \frac{\left(\sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}\right)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta \sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}}$$

$$\Rightarrow \pi(\theta|\underline{x})_j \sim Gamma\left(n, \sum_{i=1}^{n} ln(1-e^{-\lambda x_i})^{-1}\right) \tag{10}$$

Then, the posterior for Modified Jeffery prior can be found by solving the equation below.

$$\pi(\theta|\underline{x})_{MJ} = \frac{\prod_{i=1}^{n} f(x_i;\theta).g(\theta)}{\int \prod_{i=1}^{n} f(x_i;\theta).g(\theta)}$$

$$= \frac{\theta^n \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(1-e^{-\lambda x_i})^{\theta-1} \frac{1}{\sqrt[3]{\theta}}}{\int_0^\infty \theta^n \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \prod_{i=1}^{n}(1-e^{-\lambda x_i})^{\theta-1} \frac{1}{\sqrt[3]{\theta}} \quad d\theta}$$

$$= \frac{\theta^{n-\frac{3}{2}} e^{-\theta \sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}}}{\int_0^\infty \theta^{n-\frac{3}{2}} e^{-\theta \sum_{i=1}^{n} \ln(1-e^{-\lambda x_i})^{-1}} d\theta}$$

$$\pi(\theta|\underline{x})_{MJ} = \frac{\left(\sum_{i=1}^{n} \ln(^{1-e^{-\lambda x_i}})^{-1}\right)^{n-\frac{1}{2}}}{\Gamma\,(n-\frac{1}{2})}\theta^{n-\frac{3}{2}} e^{-\theta \sum_{i=1}^{n} \ln(^{1-e^{-\lambda x_i}})^{-1}}$$

$$\Rightarrow \pi(\theta|\underline{x})_{MJ} \sim Gamma\ (\,n-\tfrac{1}{2}\ \ , \sum_{i=1}^{n} ln(^{\,1-e^{-\lambda x_i}})^{-1}\,) \tag{11}$$

2.Loss Function

Based on various symmetric loss functions, squared error and precautionary, Bayes estimators of shape parameter θ are determined. In this case, the asymmetric loss function must be considered.

a) The Reliability Function Under the Squared Loss Function

One of the most famous loss functions in decision theory is the squared error loss function. It was proposed by Legendre in 1805 and Gauss in 1810 to develop the least square theory. Later, the squared error loss function is defined as: [12]

$$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2 \tag{12}$$

The Bayes estimator under the squared error loss function can be derived by using the following equation:

$$\hat{\theta}s = E(\theta \backslash \underline{X}) \tag{13}$$

The posterior density function, which was given in Equation (10) is distributed as gamma distribution with parameter$(n, \sum ln(1-e^{-\lambda x})^{-1})$. That means,

$$\pi(\theta|\underline{x})_J \approx \text{Gamma}\ \left(n, \sum ln(1-e^{-\lambda x})^{-1}\right)$$

The reliability function for Jeffrey's prior $\hat{R}(\mathrm{x})_{SJ}$ is given as

$$\hat{R}(\mathrm{x})_{SJ} = E(R(\underline{X})/\underline{x})_J \ = \int_0^{\infty} \left(\left(1-(1-e^{-\lambda x})\right)^{\theta}\right) \pi(\theta\,|\underline{x})_J\ .d\theta$$

$$= \int_0^{\infty} 1-(1-e^{-\lambda x})^{\theta} \frac{\left(\sum ln(1-e^{-\lambda x})^{-1}\right)^n}{\Gamma(n)} \theta^{n-1}\, e^{-\theta \sum ln(1-e^{-\lambda x})^{-1}} d\theta$$

$$= 1 - \left(\frac{\left(\sum ln\left(1 - e^{-\lambda x}\right)^{-1}\right)^n}{\Gamma(n)} . \int_0^{\infty} \theta^{n-1}\, e^{-\theta\left[\sum ln\left(1-e^{-\lambda x}\right)^{-1} + ln\left(1-e^{-\lambda x}\right)^{-1}\right]}\, d\theta\right)$$

$$= 1 - \left(\frac{\left(\sum ln\left(1 - e^{-\lambda x}\right)^{-1}\right)^n}{\Gamma(n)} . \frac{\Gamma(n)}{(\sum ln(1 - e^{-\lambda x})^{-1} + +ln(1 - e^{-\lambda x})^{-1})^n}\right)$$

$$= 1 - \left(\frac{\left(\sum ln\left(1 - e^{-\lambda x}\right)^{-1}\right)^n}{(\sum ln(1 - e^{-\lambda x})^{-1} + +ln(1 - e^{-\lambda x})^{-1})^n}\right)$$

$$= 1 - \left(\frac{\sum ln\left(1 - e^{-\lambda x}\right)^{-1} + ln\left(1 - e^{-\lambda x}\right)^{-1}}{\sum ln(1 - e^{-\lambda x})^{-1}}\right)^{-n}$$

$$\hat{R}(X)_{SJ} = 1 - \left(1 + \frac{ln\left(1-e^{-\lambda x}\right)^{-1}}{\sum ln\left(1-e^{-\lambda x}\right)^{-1}}\right)^{-n} \tag{14}$$

Now, the posterior density function, which was given in Equation (11) is distributed as a gamma distribution with parameter$\left(n - 1/2, \sum ln\left(1 - e^{-\lambda x}\right)^{-1}\right)$. That is,

$$\pi(\theta|\underline{x})_{MJ} \approx \text{Gamma}\left(n - \frac{1}{2}, \sum ln\left(1 - e^{-\lambda x}\right)^{-1}\right).$$

Similarly, the reliability of the Modified Jeffrey's prior $\hat{R}(\mathrm{t})_{SMJ}$ is given as:

$$\hat{R}(X)_{SMJ} = 1 - \left(1 + \frac{ln\left(1-e^{-\lambda x}\right)^{-1}}{\sum ln\left(1-e^{-\lambda x}\right)^{-1}}\right)^{-\left(n-\frac{1}{2}\right)} \tag{15}$$

b) The Reliability Function Under the Precautionary Loss Function

Norstrom offered a comprehensive class of precautionary loss functions in addition to an alternate asymmetric precautionary loss function, [13]. When underestimating might have major repercussions, these estimators are quite helpful. It is an extremely practical and straight forward $\hat{\theta}$ asymmetric precautionary loss function,

$$L\left(\hat{\theta}, \theta\right) = \frac{\left(\hat{\theta}, \theta\right)^2}{\hat{\theta}} \tag{16}$$

$$\mathrm{E}\left(\theta^2/\underline{X}\right)_J = var\left(\theta/\underline{X}\right)_J + \mathrm{E}\left(\theta/\underline{X}\right)_J$$

$$\mathrm{E}\left(\theta^2/\underline{X}\right)_J = \frac{n}{(\sum \ln(1-e^{-\lambda x})^{-1})^2} + \left(\frac{n}{(\sum \ln(1-e^{-\lambda x})^{-1}}\right)^2$$

$$\mathrm{E}\left(\theta^2/\underline{X}\right)_J = \frac{n}{(\sum \ln(1-e^{-\lambda x})^{-1})^2} + \left(\frac{n}{(\sum \ln(1-e^{-\lambda x})^{-1}}\right)^2$$

$$\Rightarrow E\left(\theta^2/\underline{X}\right)_J = \frac{n(n+1)}{(\sum ln(1-e^{-\lambda x})^{-1})^2} \tag{17}$$

Then,

$$\hat{\theta}_{PJ} = \frac{\sqrt{n(n+1)}}{(\sum ln(1-e^{-\lambda x})^{-1})} \tag{18}$$

The Bayesian estimation of θ under this asymmetric loss function is denoted by $\hat{\theta}_{\mathrm{p}}$, and it may be obtained by solving the following equation:

$$\hat{\theta}_p^2 = E(\theta^2/\underline{X}) \Rightarrow \hat{\theta}_P = \sqrt{\mathrm{E}\left(\theta^2/\underline{X}\right)}$$

By using the same loss function and posterior, the Bayes estimator of the reliability function for Jeffrey's prior $\hat{R}(x)_{PJ}$ is given as:

$$\hat{R}(x)_{PJ} = \sqrt{E\left(R\left(\underline{X}\right)^2/\underline{x}\right)_J}$$

Now,

$$E\left(R\left(\underline{X}\right)^2/\underline{x}\right)_J = \int_0^\infty \left(\left(1-\left(1-e^{-\lambda x}\right)\right)^\theta\right)^2 \pi(\theta\,|\underline{x})_J\, d\theta$$

$$= \int_0^\infty 1-2\left(1-e^{-\lambda x}\right)^\theta + (\left(1-e^{-\lambda x}\right)^\theta)^2\, \pi(\theta\,|\underline{x})_J\, d\theta$$

$$= \int_0^\infty 1-2\left(1-e^{-\lambda x}\right)^\theta + (\left(1-e^{-\lambda x}\right)^\theta)^2 \frac{\left(\sum ln\left(1-e^{-\lambda x}\right)^{-1}\right)^n}{\Gamma(n)} \theta^{n-1}\, e^{-\theta \sum ln\left(1-e^{-\lambda x}\right)^{-1}} d\theta$$

Therefore,

$$\hat{R}(X)_{PJ=}\sqrt{1-2\left(\frac{\sum ln\left(1-e^{-\lambda x}\right)^{-1}}{\sum ln\left(1-e^{-\lambda x}\right)^{-1}+ln\left(1-e^{-\lambda x}\right)^{-1}}\right)^{n}+\left(\frac{\sum ln\left(1-e^{-\lambda x}\right)^{-1}}{\sum ln\left(1-e^{-\lambda x}\right)^{-1}+2ln\left(1-e^{-\lambda x}\right)^{-1}}\right)^{n}} \quad (19)$$

Similarly, the reliability function for the modified Jeffrey's prior $\hat{R}(x)_{PMJ}$ are given as

$$\hat{R}(X)_{PMJ=}\sqrt{1-2\left(\left(\frac{\sum ln\left(1-e^{-\lambda x}\right)^{-1}}{\sum ln\left(1-e^{-\lambda x}\right)^{-1}+ln\left(1-e^{-\lambda x}\right)^{-1}}\right)^{n-\frac{1}{2}}+\left(\frac{\sum ln\left(1-e^{-\lambda x}\right)^{-1}}{\sum ln\left(1-e^{-\lambda x}\right)^{-1}+2ln\left(1-e^{-\lambda x}\right)^{-1}}\right)^{n-\frac{1}{2}}} \quad (20)$$

c) Simulation and Results

This section introduces the simulation research that was done to evaluate the shape parameter's statistical performance. The simulation is done by using Mat lab.

Four fundamental and significant steps are included in the simulation to estimate the reliability function of the generalized exponential.

First Stage:

The first stage is the most crucial since it sets the tone for the subsequent stages. Choosing the default values (true values) for the general exponential distribution parameter (λ, θ) is the first step. To investigate the impact of the parameter on the estimators when $\lambda > \theta, \lambda = \theta$, and $\lambda < \theta$. The default values are changed. By factoring in the sample size $(n = 15, 30, 50, 100)$ and the number of replicated samples (L=1000), we evaluate the values $(\lambda = 0.5, 1)$ with (θ = 0.5, 1, 1.5, 2).

Second Stage:

This stage is the data generation stage, where uniform distribution (U) is used to produce random data for the period (0, 1). The produced data is then transformed from a uniform to data distribution as generalized Exponential with shape parameters λ and θ. That is,

$$x_i = F^{-1}(U_i) = \frac{(\ln(1-U^{1/\theta}))^{-1}}{\lambda}, \text{where } i \geq 1. \quad (21)$$

Third stage:

Using the comparison scale (MSE) according to the formula below, real and estimator values of the different methods are compared at this stage of the research.

$$MSE(\widehat{\theta}) = \frac{\sum_{i=0}^{n}(\hat{\theta}_L - \theta)^2}{L} , MSE(\hat{R}(t)) = \frac{\sum_{i=0}^{n}(\hat{R}(t)_L - R(t))^2}{L} \quad (22)$$

Fourth stage:-

As indicated in the tables below, the simulation results are provided at this stage to determine the best estimation of the parameter of the form and the function of the dependencies.

Table 1: The Estimated value of the Reliability function and MSE values for different estimation methods. When $\theta = 0.5,\ \lambda = 0.5, and\ R = 1000.$

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.373	0.430	0.444	0.169	0.236	0.999	0.881	$\hat{R}PES$
	1.5	0.205	0 .009	0.659	0.262	0.352	0.049	0.181	
	2	0.118	0.808	0.219	0.322	0.421	0.028	0.001	
	2.5	0.155	0.446	0.612	0.375	0.476	0.016	0.1037	
	3	0.205	0.212	0.811	0.424	0.518	0.009	0.425	
	MES		0.999	0 .009	0.028	0.444	0.659	0.446	
30	1	0.373	0169	0.447	0.131	0.131	0.450	0.233	$\hat{R}PES$
	1.5	0.205	0.079	0.884	0.174	0.176	0.729	0.506	
	2	0.118	0.331	0.459	0.209	0.211	0.591	0.278	
	2.5	0.155	0.601	0.146	0.236	0.238	0.967	0.730	
	3	0.205	0.542	0.053	0.258	0.259	0.353	0.318	
	MES		0.292	0.217	0.515	0.515	0.411	0.062	
50	1	0.373	0.075	0.539	0.078	0.079	0.161	0.796	$\hat{R}PES$
	1.5	0.205	0.139	0.531	0.106	0.108	0.502	0.667	
	2	0.118	0.080	0.778	0.126	0.126	0.173	0.907	
	2.5	0.155	0.104	0.693	0.143	0. 143	0.304	0.525	
	3	0.205	0.292	0.217	0.515	0.515	0.004	0.062	
	MES		0.396	0009	0.025	0.023	0.150	0. 014	
100	1	0.373	0.040	0.001	0.050	0.040	0.454	0.099	$\hat{R}PES$
	1.5	0.205	0.512	0.282	0.053	0.053	0.316	0.025	
	2	0.118	0.062	0.010	0.063	0.063	0.354	0.972	
	2.5	0.155	0.082	0.030	0.071	0.071	0.724	0.717	
	3	0.205	0.071	0.004	0.078	0.777	0.045	0.153	
	MES		0.0971	0.032	0.054	0.050	0.050	0.738	

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Table 2: The Estimated value of the Reliability function and MSE values for different estimation methods. When θ=1,λ=0.5, and R=1000

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.607	0.375	0.211	0.077	0.074	0.433	0.150	$\hat{R}PES$
	1.5	0.472	0.984	0.026	0.143	0.139	0.306	0.121	
	2	0.368	0.106	0.0002	0.215	0.210	0.137	0.323	
	2.5	0.287	0.377	0.002	0.287	0.281	0.178	0.434	
	3	0.223	0.100	0.005	0.354	0.348	0.676	0.556	
	MES		0.215	0.074	0.121	0.211	0.281	0.984	
30	1	0.607	0.1526	0.122	0.051	0.051	0.947	0.471	$\hat{R}MLE$
	1.5	0.472	0.266	0.0918	0.091	0.09132	0.436	0.05	
	2	0.368	0.388	0.427	0.135	0.131	0.393	0.098	
	2.5	0.287	0.238	0.184	0.168	0.168	0.379	0.111	
	3	0.223	0.888	0.064	0.201	0.200	0.582	0.565	
	MES		0.007	0.397	0.0367	0.0154	0.015	0.5504	
50	1	0.607	0.397	0.0367	0.0154	0.015	0.5504	0.214	$\hat{R}PES$
	1.5	0.472	0.748	0.044	0.056	0.057	0.0679	0.064	
	2	0.368	0.010	0.027	0.080	0.079	0.150	0.237	
	2.5	0.287	0.003	0.016	0.102	0.109	0.333	0.128	
	3	0.223	0.517	0.009	0.121	0.121	0.110	0.5759	
	MSE		0.343	0.021	0.139	0.156	0.156	0.582	
100	1	0.607	0.397	0.0367	0.0154	0.015	0.5504	0.045	$\hat{R}PES$
	1.5	0.472	0.276	0.022	0.0279	0.028	0.0741	0.078	
	2	0.368	0.256	0.013	0.030	0.04	0.1044	0.103	
	2.5	0.287	0.142	0.008	0.051	0.051	0.474	0.1022	
	3	0.223	0.105	0.005	0.0607	0.060	0.4128	0.0995	
	MSE		0.3973	0.037	0.306	0.128	0.134	0.120	

Table 3: The Estimated value of the Reliability function and MSE values for different estimation methods When $\theta = 1.5, \lambda = 0.5,$ and $R = 1000$

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.7532	0.396	0.349	0.025	0.023	0.150	0. 014	$\hat{R}PMJ$
	1.5	0.6167	0.143	0.083	0.068	0.064	0.064	0.002	
	2	0.4974	0.990	0.007	0.129	0.124	0.125	0.004	
	2.5	0.3973	0.261	0.114	0.198	0.193	0.178	0.041	
	3	0.3153	0.044	0.004	0.207	0.265	0.564	0.009	
	MSE		0.331	0.459	0.209	0.211	0.591	0.028	
30	1	0.7532	0.067	0.009	0.020	0.020	0.027	0.375	$\hat{R}PES$
	1.5	0.6167	0.822	0.248	0.048	0.048	0.437	0.986	
	2	0.4974	0.429	0.462	0.083	0.0826	0.393	0.106	
	2.5	0.3973	0.403	0.306	0.12	0.134	0.120	0.377	
	3	0.3153	0.895	0.139	0.156	0.156	0.582	0.1003	
	MSE		0.065	0.0219	0.173	0.173	0.478	0.0390	
50	1	0.7532	0.728	0.111	0.012	0.0122	0.265	0.131	$\hat{R}PES$
	1.5	0.6167	0.845	0.075	0.029	0.034	0.068	0.174	
	2	0.4974	0.169	0.049	0.050	0.050	0.142	0.412	

100	2.5	0.3973	0.037	0.0312	0.073	0.073	0.333	0.206	
	3	0.3153	0.624	0.0196	0.094	0.094	0.110	0.463	
	MSE		0.248	0.111	0.112	0.112	0.478	0.462	$\hat{R}PES$
	1	0.7532	0.802	0.056	0.006	0.006	0.550	0.074	
	1.5	0.6167	0.979	0.038	0.015	0.015	0.074	0.139	
	2	0.4974	0.372	0.016	0.036	0.036	0.104	0.210	
	2.5	0.3973	0.372	0.016	0.036	0.036	0.747	0.281	
	3	0.3153	0.076	0.002	0.047	0.049	0.412	0.349	
	MSE		0.215	0.050	0.056	0.056	0.709	0.124	

Table 4: The Estimated value of the Reliability function and MSE values for different estimation methods. When $\theta = 2, \lambda = 0.5,$ and $R = 1000$

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.8452	0.479	0.371	0.007	0.0055	0.2150	0.0562	$\hat{R}PMJ$
	1.5	0.7216	0.262	0.148	0.03	0.024	0.3063	0.0377	
	2	0.6004	0.879	0.024	0.0748	0.073	0.1365	0.0150	
	2.5	0.4909	0.176	0.0002	0.1357	0.132	0.1177	0.0157	
	3	0.3965	0.014	0.003	0.207	0.201	0.674	0.009	
	MSE		0.080	0.778	0.126	0.126	0.173	0.003	
30	1	0.8452	0.039	0.009	0.008	0.0079	0.9470	0.006	$\hat{R}PMJ$
	1.5	0.7216	0.474	0.037	0.025	0.020	0.4364	0.0145	
	2	0.6004	0.491	0.328	0.052	0.0521	0.3929	0.036	
	2.5	0.4909	0.518	0.354	0.0853	0.0859	0.3788	0.036	
	3	0.3965	0.874	0.207	0.1200	0.120	0.5822	0.0468	
	MSE		0.229	0.215	0.213	0.213	0.151	0.149	
50	1	0.8452	0.039	0.009	0.008	0.018	0.947	0.210	$\hat{R}SJ$
	1.5	0.7216	0.474	0.037	0.025	0.025	0.435	0.256	
	2	0.6004	0.491	0.329	0.052	0.052	0.393	0.285	
	2.5	0.4909	0.081	0.048	0.059	0.052	0.333	0.305	
	3	0.3965	0.723	0.031	0.073	0.073	0.110	0.316	
	MSE		0.076	0.169	0.169	0.019	0.056	0.287	
100	1	0.8452	0.341	0.071	0.002	0.002'	0.550	0.210	$\hat{R}SJ$
	1.5	0.7216	0.621	0.052	0.008	0.008	0.074	0.329	
	2	0.6004	0.229	0.035	0.017	0.016	0.104	0.428	
	2.5	0.4909	0.857	0.024	0.026	0.028	0.747	0.503	
	3	0.3965	0.035	0.016	0.036	0.036	0.413	0.554	
	MSE		0.808	0.219	0.322	0.004	0.028	0.011	

Table 5: The Estimated value of the Reliability function and MSE values for different estimation methods. When θ =0.5, λ=1, and R=1000

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.2049	0.037	0.421	0.350	0.343	0.137	0.397	$\hat{R}MLE$
	1.5	0.1186	0.118	0.518	0.435	0.427	0.676	0.276	
	2	0.0701	0.077	0.576	0.050	0.493	0.108	0.256	
	2.5	0.0419	0.032	0.612	0.551	0.545	0.651	0.168	
	3	0.0252	0.014	0.633	0.590	0.585	0.128	0.105	
	MSE		0.0433	0.106	0.072	0.069	0.135	0.066	
30	1	0.2049	0.942	0.211	0.211	0.210	0.393	0.047	$\hat{R}MLE$
	1.5	0.1186	0.163	0.259	0.256	0.256	0.583	0.047	
	2	0.0701	0.036	0.288	0.286	0.285	0.151	0.103	
	2.5	0.0419	0.037	0.306	0.305	0.305	0.099	0.102	
	3	0.0252	0.0181	0.317	0.317	0.316	0.105	0.094	
	MSE		0.034	0.053	0.258	0.259	0.353	0.542	
50	1	0.2049	0.278	0.126	0.126	0.127	0.149	0.015	$\hat{R}PMJ$
	1.5	0.1186	0.622	0.165	0.155	0.155	0.110	0.0278	
	2	0.0701	0.065	0.219	0.173	0.173	0.478	0.0390	
	2.5	0.0419	0.019	0.165	0.184	0.184	0.056	0.051	
	3	0.0252	0.0202	0.079	0.190	0.190	0.398	0.0601	
	MSE		0.136	0.688	0.986	0.285	0.191	0.011	
100	1	0.2049	0.126	0.073	0.0632	0.063	0.104	0.015	$\hat{R}MLE$
	1.5	0.1186	0.814	0.096	0.0777	0.078	0.412	0.028	
	2	0.0701	0.081	0.062	0.087	0.086	0.709	0.040	
	2.5	0.0419	0.220	0.220	0.092	0.817	0.817	0.050	
	3	0.0252	0.003	0.01	0.095	0.095	0.150	0.060	
	MSE		0.0312	0.282	0.053	0.053	0.316	0.512	

Table 6: The Estimated value of the Reliability function and MSE values for different estimation methods. When $\boldsymbol{\theta}$=1, λ=1, and R=1000

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.3679	0.354	0.266	0.210	0.205	0.137	0.550	$\hat{R}MLE$
	1.5	0.2231	0.352	0.402	0.329	0.676	0.676	0.074	
	2	0.1353	0.144	0.498	0.428	0.421	0.108	0.104	
	2.5	0.0821	0.053	0.562	0.503	0.497	0.651	0.472	
	3	0.049	0.019	0.602	0.554	0.554	0.128	0.413	
	MSE		0.215	0.056	0.056	0.056	0.709	0.124	
30	1	0.3679	0.463	0.133	0.133	0.013	0.392	0.031	$\hat{R}MLE$
	1.5	0.2231	0.055	0.203	0.199	0.198	0.582	0.056	
	2	0.1353	0.150	0.249	0.247	0.245	0.151	0.08	
	2.5	0.0821	0.071	0.281	0.280	0.279	0.099	0.102	
	3	0.049	0.028	0.301	0.301	0.301	0.105	0.121	
	MSE		0.0413	0.106	0.072	0.069	0.135	0.066	

50	1	0.3679	0.754	0.081	0.080	0.080	0.149	0.077	$\hat{R}MLE$
	1.5	0.2231	0.171	0.121	0.121	0.121	0.110	0.143	
	2	0.1353	0.033	0.149	0.151	0.149	0.478	0.215	
	2.5	0.0821	0.076	0.169	0.169	0.169	0.056	0.287	
	3	0.049	0.037	0.181	0.181	0.180	0.398	0.354	
	MSE		0.0276	0.122	0.3279	0.428	01741	0.978	
100	1	0.3679	0.757	0.020	0.020	0.04	0.104	0.051	$\hat{R}MLE$
	1.5	0.2231	0.501	0.060	0.060	0.060	0.413	0.091	
	2	0.1353	0.027	0.075	0.075	0.074	0.709	0.131	
	2.5	0.0821	0.002	0.084	0.084	0.084	0.817	0.168	
	3	0.049	0.035	0.090	0.090	0.090	0.151	0.201	
	MSE		0.0372	0.016	0.036	0.036	0.104	0.210	

Table 7: The Estimated value of the Reliability function and MSE values for different estimation methods. When $\boldsymbol{\theta}$=1.5, λ=1, and R=1000.

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
5	1	0.4974	0.944	0.168	0.125	0.120	0.137	0.717	$\hat{R}PMJ$
	1.5	0.3152	0.522	0.313	0.249	0.242	0.759	0.748	
	2	0.196	0.196	0.432	0.365	0.359	0.08	0.012	
	2.5	0.1206	0.071	0.516	0.46	0.454	0.509	0.003	
	3	0.0737	0.026	0.572	0.531	0.522	0.289	0.517	
	MSE		0.215	0.456	0.556	0.256	0.709	0.124	
30	1	0.4974	0.240	0.084	0.084	0.084	0.393	0 .265	$\hat{R}PMJ$
	1.5	0.3152	0.289	0.156	0.154	0.154	0.583	0.067	
	2	0.196	0.229	0.215	0.213	0.213	0.151	0.149	
	2.5	0.1206	0.092	0.258	0.257	0.256	0.099	0.333	
	3	0.0737	0.034	0.286	0.286	0.286	0.105	0.113	
	MSE		0.352	0.402	0.329	0.676	0.676	0.074	
50	1	0.4974	0.193	0.025	0.025	0.025	0.104	0.947	$\hat{R}SJ$ $+ \hat{R}SMJ$
	1.5	0.3152	0.721	0.047	0.047	0.047	0.413	0.4364	
	2	0.19597	0.775	0.065	0.067	0.067	0.709	0.392	
	2.5	0.12057	0.028	0.077	0.077	0.077	0.816	0.379	
	3	0.0737	'0.059	0.086	0.085	0.085	0.150	0.582	
	MSE		0.852	0.402	0.402	0.676	0.676	0.674	
100	1	0.4974	0.931	0.025	0.025	0.025	0.043	0.211	$\hat{R}PMJ$
	1.5	0.3152	0.72	0.047	0.047	0.047	0. 127	0.026	
	2	0.196	0.775	0.065	0.065	0.065	0.709	0.001	
	2.5	0.1206	0.028	0.077	0.077	0.077	0.817	0.002	
	3	0.0737	0.057	0.086	0.086	0.086	0.150	0.004	
	MSE		0.035	0.090	0.090	0.090	0.151	0.0201	

Table 8: The Estimated value of the Reliability function and MSE values for different estimation methods. When $\boldsymbol{\theta}$=2, λ=1, and R=1000

N	x_i	Real R(t)	$\hat{R}MLE$	$\hat{R}PES$	$\hat{R}SJ$	$\hat{R}SMJ$	$\hat{R}PJ$	$\hat{R}PMJ$	Best
15	1	0.6004	0.433	0.106	0.072	0.069	0.135	0.066	$\hat{R}MLE$
	1.5	0.3965	0.660	0.243	0.190	0.182	0.6759	0.821	
	2	0.2524	0.243	0.373	0.31187.	0.307	0.0820	0.429	
	2.5	0.1574	0.089	0.473	0.420	0.414	0.509	0.402	
	3	0.0971	0.032	0.054	0.050	0.050	0.028	0.070	
	MSE		0.044	0.248	0.890	0.482	0.975	0.082	
30	1	0.6004	0.325	0.057	0.053	0.053	0.393	0.375	$\hat{R}SJ + \hat{R}PMJ$
	1.5	0.3965	0.517	0.121	0.119	0.582	0.582	0.986	
	2	0.2524	0.287	0.186	0.184	0.184	0.151	0.106	
	2.5	0.1574	0.109	0.237	0.235	0.099	0.099	0.377	
	3	0.0971	0.039	0.272	0.271	0.271	0.105	0.100	
	MSE		0.095	0.190	0.090	0.190	0.151	0.090	
50	1	0.6004	0.882	0.032	0.032	0.149	0.149	0.009	$\hat{R}SJ + \hat{R}PMJ$
	1.5	0.3965	0.164	0.073	0.073	0.073	0.110	0.248	
	2	0.2524	0.248	0.115	0.112	0.112	0.478	0.462	
	2.5	0.1574	0.136	0.142	0.142	0.142	0.0557	0.307	
	3	0.0971	0.052	0.1630	0.163	0.163	0.398	0.139	
	MSE		0.352	0.402	0.074	0.676	0.676	0.074	
100	1	0.6004	0.014	0.016	0.016	0.016	0.104	0.023	$\hat{R}PES + \hat{R}SJ + \hat{R}PMJ$
	1.5	0.3965	0.456	0.036	0.036	0.036	0.413	0.064	
	2	0.2524	0.215	0.056	0.056	0.056	0.709	0.124	
	2.5	0.1574	0.081	0.071	0.071	0.071	0.817	0.194	
	3	0.0971	0.073	0.082	0.082	0.082	0.1500	0.263	
	MSE		0.352	0.111	0.111	0.676	0.676	0.111	

Conclusions

1. In Tables 1,2 and 3when ($\theta = 0.5, 1, 1.5, \lambda = 0.5$) the reliability function for the (PEC) is the best because it has less MSE compared with other methods for the larger sample size.
2. In Table 4, when ($\theta = 2, \lambda = 0.5$) the reliability function is for the (SJ, PMJ) is the best because it has less MSE compare with other methods for a larger sample size.
3. In the Table 5 and 6, the MSE values were different and the larger sample size was the preference for the methods (MLE).
4. In Table 7 and 8, the MSE values were different and the larger sample size was the preference for the methods (PMJ and SJ).

Recommendations

1. Use different loss functions to get more estimators for the reliability function R(t)
2. Using a different scale to compare the reliability function R(t) and the best estimators.

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