Academic Science Journal

Fuzzy Semi Injective Subact

Arbah Sultan Abdul Kareem and Yusra kareem Abd

Department of Mathematics-College of Science, University of Diyala

yusrakareem77@gmail.com

Received: 11 May 2023 **Accepted: 31 July 2023**

DOI: https://dx.doi.org/10.24237/ASJ.02.03.760B

Abstract

In this paper we introduced the notion of the subact of FS-injective S-act, we gave an example show that subact of FS-injective S-acts need not be FS-injective. Thus we investigated some kinds of sub acts which inherit the FS-injective property such as fuzzy fully invariant subact , fuzzy retract subact ,fuzzy intersection-large and fuzzy stable subact finally we studied some properties of the subacts of FS-injective .

Keyword: Semi injective S-act, Fuzzy Injective, fuzzy quasi injective, fuzzy semi injective.

الضبابية لشبه التباين الجزئي

ارباح سلطان عبد الكريم و يسرى كريم عبد

قسم الرياضيات - كلية العلوم - جامعة ديالى

الخلاصة

في هذا البحث نحن نقدم مفهوم ضبابية شبه المتباينة الجزئي بإعطاء مثال يبين أن النظام الجزئي من شبه المتباينة الضبابي ليست من الضروري ان يكون ضبابي ايضا وعليه نحن حققنا بعض الانواع الجزئية من شبه التباين الضبابي التي ترث صفة الضبابية ومن امثلتها الانكماش الضبابي الجزئي ,التقاطع الكبير الضبابي الجزئي ,الثابت التام الضبابي الجزئي والثابت الضبابي الجزئي واخيرا نحن درسنا بعض خواص شبه التباين الضبابي الجزئي .

كلمات مفتاحية: شبه التباين للنظام S, التباين الضبابي, شبه التباين الضبابي ونصف التباين الضبابي .

Introduction

Injective and quasi injective S-act were studied by [4,8], after that [5] initiated semi-injective S-act as a generalization of quasi injective S-act and semi injective module. In application point of view, fuzzy S-act was widely used in many engineering applications such as model fuzzy discrete system .This has been giving the motivation in recent years to investigate fuzziness of different kinds of S-acts.

In (2021) fuzzy injective and fuzzy quasi injective S-acts were defined and studied by many others (see, [1,7]). As a generalization of fuzzy quasi injective S-act and fuzzy semi injective module [2]. Fuzzy semi injective S-act (FS-injective) has been introduced by [6].

In this research we investigated the fuzzy sub acts of FS-injective S-act .First we showed by examples that fuzzy subact of FS-injective need not be FS-injective S-act, so we gave some kinds of subacts that inherits the fuzzy semi –injectivity property and studied some of their properties .

Preliminaries

Definition (1) [9]: An S-homomorphism f:A→B is called retraction if there exist an S-homomorphism h:B→A such that $f \circ h = I_B$ and B said to be retract of A.

Definition (2)[5]: An S-act N is said to be multiplication if each subact of N has the form NI for some right ideal I of S. For example Z is a multiplication (Z,.)-act.

Definition(3)[5]: Anon zero subact H of an S-act N is called an intersection large (∩-large) if for all nonzero –act X of N, X∩H≠0 this will denoted by $H\subseteq' N$.

Definition (5)[8]: An S-act N is quasi-injective if for any subact K of N and g∈hom(K,N) there exist an S-homomorphism h: N→N such that $h|_k$=g.

Definition (6)[5]: An S-act N is said to semi injective if for each subact H of N and S-monomorphism g:H→H and each S-homomorphism k:H→H ; there exist an S-homomorphism f: N→N such that f ∘ inc ∘ g = inc ∘g ∘ k.

Academic Science Journal

Definition (7)[6]: Let λ be a fuzzy S-act on the S-act L then it is called fuzzy semi-injective(FS-injective)If :

1. L is semi-injective S-act.
2. λ (ℓ) ≤ λ(f(ℓ)) ∀f ∈hom(L,L)

Definition (8)[7]: Let B be a fully invariant and if (B, λ_B)is a fuzzy subact of fuzzy S-act (L, λ_L) then(B, λ_B) is called fuzzy fully invariant β(λ_B)⊆ λ_B for each fuzzy S-homomorphism .

β: (L, λ_L)⟶ (L, λ_L).

Definition(9)[7]: Let A be a stable and let (A, λ_A) be a fuzzy subact of fuzzy S-act (L, λ_L)then (A, λ_A)Is called fuzzy stable if $\varphi(\lambda_A) \subseteq \lambda_A$ for each fuzzy S-homomorphism φ: (A, λ_A) →(L, λ_L) .

Definition (10)[7] : Let (H, λ_H) and (L, λ_L) be two fuzzy S-acts then (H, λ_H) is called fuzzy retract of (L, λ_L) if there exist a fuzzy S-homomorphism Φ:(L, λ_L)⟶ (H, λ_H) and ψ: (H, λ_H) ⟶(L, λ_L) such that ψ ∘ ϕ=I_L .

Definition (11)[7]: Anon –zero fuzzy subact (H,λ_H) of fuzzy S-act (N,λ_N) is called fuzzy intersection- large if for all non-zero fuzzy subact (B,λ_B.) of λ_N such that $\lambda_B \wedge \lambda_H \neq 0$.

Definition (12)[7]: Let L be a quasi –injective S-act (L,λ_L) is called fuzzy quasi injective S-act (FQ-injective)if for each fuzzy subact(H, λ_L) of (L,λ_L) and for each fuzzy S-homomorphism ψ: $(H, \lambda_H) \rightarrow (L, \lambda_L)$ there exist a fuzzy S-homomorphism ϕ: $(L, \lambda_L) \rightarrow (L, \lambda_L)$ such that ϕ is an extension of ψ .

Proposition(13)[5] : Every subact of multiplication of semi-injective S-act is semi- injective .

Proposition (14) [5]: Every stable subact of semi-injective is semi-injective .

Proposition(15)[5]: Let N be a semi-injective S- act then every ∩-large quasi –injective subact of N is stable .

Proposition (16)[5]: Every fully invariant subact of semi-injective S-act is semi – injective .
Proposition (17)[5]: Every ∩-large fully stable subact of semi –injective S-act is quasi-injective .

Fuzzy semi-injective subact

In this section, we will investigate the subacts of FS-injective

Definition(1) :Let H be a subact of the act L, λ_H and λ_L are the characteristic functions of H and L respectively, suppose that (L , λ_L) is FSI-act if (H, λ_H)is a FSI- act by itself then (H, λ_H) is called fuzzy semi-injective S-subact of (L, λ_L)(FS-injective subact) .

In the following examples we will show that a subact of FS-injective need not be FS-injective.

Examples(2): 1. Let M=Z⊕Q be an Z-act if λ_s is any fuzzy set on M then it is easy matter to show that (M, λ_M) is not FS- injective subact of the FS- injective S-act (E(M),$\lambda_{E(M)}$).(since it contract condition one of our definition see[3]) .

2.Let M={1,2,3,4,5,6}, then the subgroup of the permutation group S_6, G=<{(1234),(56)}> is an abelian group generated by (1234)and (56)of order 8. We see easily that G={I,(1234), (13)(24), (1432), (56), (1234)(56), (13)(24)(56),(1432)(56)} where I is identity permutation .We view that G is not semi injective Z-act . Also if we suppose M={I,(13)(24), (56), (13)(24)(56)}. Define g:M $\longrightarrow$ M by

$$g(x)=\begin{cases} (56) & x=(13)(24),(56) \\ I & x=(13)(24)(56),I \end{cases}$$

If G is semi injective Z- act, then there exist an extension f:G $\longrightarrow$G of g .Then f((1234))=a for some a∈G hence f((1234) o (1234))= f((1234)) o f((1234))=a^2. So (56) =f((13)(24))= a^2 if a is an even or odd permutation of G then a^2 is even permutation which is a contradiction in both cases (56) is odd permutation . Thus G is not semi injective sub act of M but injective envelope of G is semi injective .

Proposition (3) : Every fuzzy fully invariant subact of FS- injective S-act is FS-injective S-act .

Proof : For each fuzzy subact (K, λ_k)of a fuzzy fully invariant subact (N, λ_N) of FS-injective (M, λ_M) and each fuzzy S-homomorphism f: (K, λ_k)→ (K, λ_k) as the following figure(1) :

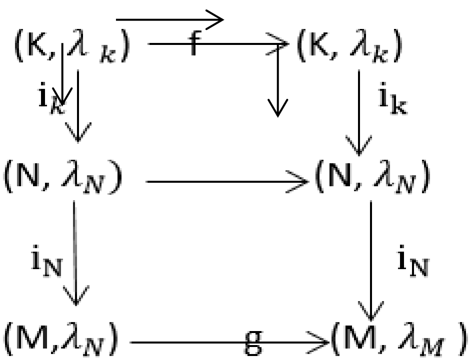

Figure 1

Wherei$_K$,i$_N$ are the inclusion mapping of N into M by FS- injective there exist g: (M, λ_M) ⟶ (M, λ_M) such that g extension of f:

Now we must prove (N, λ_N) is fuzzy semi injective:

1. N is semi injective by proposition (2.16)
2. To prove λ_N(h(N))≥ λ_N(N) ∀ h ∈ hom(N,N) .

Now ∀ h ∈ hom(N, N)

∃ g ∈ hom (M,M) such that g(N)=h we get λ_N(h(n)= λ_N(g(n) ≥ λ_N(N)

(since λ_N is fuzzy fully invariant) .

Hence λ_N(h(n))≥ λ_N(n) there fore λ_N is FS- injective.

In order to show that FS-injective property inherits for fuzzy retract we need the following:

Lemma(4) :A retract of semi-injective S-act is semi-injective .

Proof: Let N be a retract of semi injective S-act M then N is a direct summand by (if N is a subact of an S-act L then N is a retract iff N is a direct summand) .

Let A be a subact of N then there exist g$\in$ End (M) .

such that $g\circ i_N \circ i_A = i_N \circ i_A \circ f$ as in the figure(2):

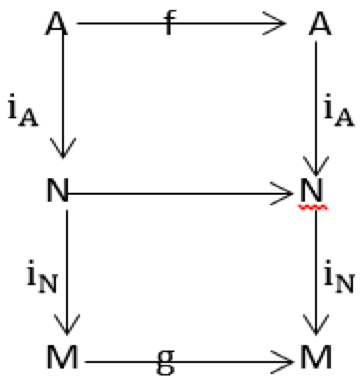

Figure 2

We can find h:N→N where $h=\pi_N \circ g \circ i_N$ where π_N is the projection map from M onto N (since N is direct summand of M)

Now $h\circ i_A=\pi_N \circ g \circ i_N \circ i_A=\pi_N \circ i_N \circ i_A \circ f \Longrightarrow h\circ i_A=i_A \circ f$

$\Longrightarrow$N is semi-injective.

Proposition (5):A fuzzy retract of FSI-act is FS- injective .

Proof: Let (C,λ_C) be a fuzzy retract of FS-injective S-act act (D, λ_D) then C is semi injective S-act by (4) there exist a fuzzy S-homomorphism f: (C,λ_C)→ (D, λ_D)

So λ_D(f(c))≥ λ_C(c) (1)

So λ_c(g(d))≥ λ_D(d) (2)

And g: (D, λ_D)$\rightarrow$ (C,λ_C)

But D is semi injective $\Longrightarrow \exists\ \alpha$:D$\rightarrow$D such that

$\lambda_D(\alpha$ (d))$\geq \lambda_D$(d) $\forall$d $\in$D (3)

And hence $\exists\ \theta \in$ End (c) where θ=g$\circ\ \alpha\ \circ$f

$\lambda_c(\theta$ (c))= λ_c (g $\circ\ \alpha\ \circ$f)(c))= λ_c (g(α(f)(c))) $\geq\lambda_D$ ((α(f)(c) by(2)

$\geq\lambda_D$ (((f)(c)) by(3)

$\geq\ \lambda_C$(c) by(1)

Thus (C,λ_C) is FS-injective S-act .

Proposition(6): Every fuzzy sub act of multiplication FS- injective S-act is FSI-act.

Proof: Let (L,λ_L) be multiplication FS- injective and let (A, λ_A) fuzzy sub act of (L, λ_L) by proposition (13) we get A is semi-injective S-act .To show $\lambda_A(\alpha$(a)) $\geq\lambda_A$(a)

$\forall\ \alpha \in$ hom(A,A) there exist β $\in$ hom(L,L) such that β= α(A)= α(IL)=I α(L)$\subseteq$ IL=A

(since (L,λ_L) multiplication) .

λ_A (α(a)= λ_A (β(a) $\geq\lambda_A$(a) thus λ_A (α(a)) $\geq\lambda_A$(a) there for λ_A is FS –injective

See figure(3):

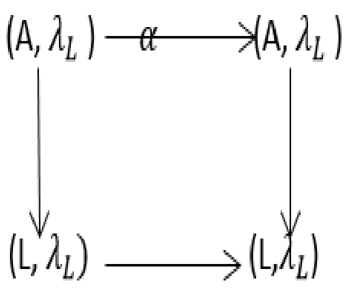

Figure 3

Proposition (7): Every fuzzy stable sub act of FS- injective S-act is FS- injective with $\lambda_L|_c = \lambda_c$.

Proof: Let (L,λ_L) be aFS –injective S-act and (C, λ_C) be a fuzzy stable sub act by (2.14) C is semi injective so there exist ψ ∈ End(C).As in the figure (4):

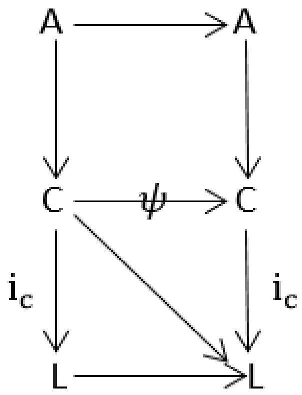

Figure 4

Let f=i_c ∘ ψ ∈hom(C,L)

λ_C (ψ(c))= $\lambda_C(i_c$ ∘ ψ(c))= λ_C(f(c)) by stability of C , f(c) ∈ C

=λ_L(f(c)≥ λ_C(c) by semi injective of L

=λ_C(c) .

Thus (C, λ_C) is FS- injective S-act .

Proposition (8) : Every intersection large fuzzy fully stable sub act of semi-injective S-act is quasi –injective .

Proof: Let(L, λ_L) be FS- injective S-act and (c, λ_C) is fuzzy fully stable subact of (L, λ_L) . (A, λ_A). be a fuzzy subact of (c, λ_C) .
Now for each fuzzy S-homomorphism f∈End (C,A)by fully stability of C we have f ∈End (c) , but (L, λ_L) is FS- injective S-act so there exist g ∈End (L) which is extension of f and λ_L(g(ℓ)))≥ $\lambda_L(\ell)$.

Academic Science Journal

Again by fuzzy fully stability of (c, λ_C) we have g$|_C$ ∈End (C) and λ_c(h(c)))≥ $\lambda_c(c)$

So C is FQ –injective.

Proposition (9): Let (L, λ_L) be FS- injective S-act then every intersection large FQ-injective subact of (L, λ_L) is fuzzy stable .

Proof: Let(A,λ_A)be an intersection large FQ- injective sub act of (L, λ_L) for every fuzzy S-homo f ∈hom (A,L) . Let A'=f^{-1}(A) and α=f $|_{A'}$ ∈hom (A',A) by FQ- injective of (A,λ_A) then ∃ g∈ End (A) , And λ_A(g(a))≥ $\lambda_A(a)$ ∀ a ∈ A by FS- injective of (L, λ_L). Then there exist h∈ End (L) And λ_L(h(x))≥ λ_L(x) ∀ x ∈ L .

But (A,λ_A) is ∩-large, So f=h$|_A$=g see the proof of proposition (2.15) .

Then λ_A(f(a))= λ_A(h(a))= λ_A(g(a)))) ≥ $\lambda_A(a)$ ⟹f(λ_A) ⊆ λ_A .

⟹(A,λ_A) is fuzzy stable.

Theorem (10)

If λ is FS- injective S-act on N and H is semi injective S-subact of N then there exist FS-injective S-act on H :

Proof: Since λ is FS- injective S-act on N then:

1.N is semi injective 2. λ(n)≤ λ(g(n)) ∀ g∈ hom (N,N)

Let V=λ_H to prove that V is FS- injective, since H is semi injective by hypothesis we must just show that for the second condition as follows

V(h)= λ(h)≤ λ(g(h)) but g$|_H$=ψ ∀ψ ∈ hom (H,H)

= λ(ψ(h))= V(ψ(h)) .

Thus V is FS- injective.

Corollary(11) : If L multiplication FS- injective then there exist FS- injective S-act on each subact of L .

Proof: Let H is a subact of L then by (13) we get H is a semi injective S-act and by theorem (3.10) there exist λ_H FS- injective on H.

Conclusion

Our goal in this article is to introduce the notion of FS-injective subact used this new ideal to show that is not necessary every subact of FS-injective S-act is FS- injective, also we gave some new kinds of subact which inherit the FS-injective property such as fuzzy fully invariant subact, fuzzy retract subact,fuzzy intersection-large and fuzzy stable sub act .

References

1. A. S. Abdul Kareem, fuzzy injective S-act over monoids,TWMS Journal of applied and engineering mathematics,11(4),1072-1079(2021)
2. A. S. Abdul Kareem, On fuzzy semi-injective modules, the Journal of fuzzy Mathematics ,26(2), 275-279(2018)
3. A. S. Abdul kareem, On semi injective modules, Master thesis of sciences in Mathematics, (1997)
4. P. Berthiam, The injective envelope of S-sets,Canad.Math.Bull,10(2), 261-273(1967)
5. M. A. A. Dahash, On semi injective S-system, international journal of algebra ,8(10), 459-470(2014)
6. A. S. Y. Kareem, On fuzzy semi injective S-act , about to appear, (2023)
7. T. F. Lateef, On some generalizations of fuzzy quasi –injective S-acts, MScs thesis, University of Diyala, (2021)
8. A. M. Lopez, Jr. K. L. ue deman, Quasi –injective S-system and their S-endomorphism semigroup, Czechoslovak, Math Journal ,29(1), 97-104(1979)
9. G. Moghaddasi, On injective and subdirectly irreducible S-acts, Over left zero semigroup, Turk Journal Math ,36(3), 359-365(2012)