Academic Science Journal

Exponential Spline Method for Solving Fuzzy Integro-Differential Equations

Fatima K. Dawood* and Rokan Khaji

Department of Mathematics, College of Science, Diyala University

* fkadum8@gmail.com

Received: 4 June 2023 **Accepted: 6 August 2023**

DOI: https://dx.doi.org/10.24237/ASJ.02.03.768B

Abstract

In this paper, we consider a new class of fuzzy functions called Fuzzy Integro- Differential Equations.Some numerical methods, such as cubic spline have been used to determine the solutions of these equations. We extend these numerical techniques to find the optimal solutions. Exponential spline technique is used for this. The results shown that Exponential spline method is more accurate in terms of absolute error. Based on the parametric form of the fuzzy number, the integro- differential equation is contacted into two systems of the second kind. Illustrative examples are given to demonstrate the high precision and good performance of the new class. Graphical representations reveal the symmetry between lower and upper cut represent of fuzzy solutions and may be helpful for a better understanding of fuzzy model artificial, intelligence, medical science and quantum.

Keywords: Fuzzy integro-differential equations, Exponential Spline, Exact solution, Approximate solution, Fuzzy parameter.

طريقة السبلاين الاسي لحل المعادلات التفاضلية التكاملية الضبابية

فاطمه كاظم داود و روكان خاجي

قسم الرياضيات- كلية العلوم- جامعة ديالى

الخلاصة

في هذا البحث ، نأخذ في الاعتبار فئة جديدة من الدوال الغامضة تسمى المعادلات التفاضلية المتكاملة الضبابية. تم استخدام بعض الطرق العددية مثل دالة السبلاين التكعيبية لتحديد حلول هذه المعادلات. نقوم بتوسيع هذه التقنيات العددية لإيجاد الحلول المثلى. يتم استخدام تقنية السبلاين الأسي لهذا الغرض. أظهرت النتائج أن طريقة السبلاين الأسي

أكثر دقة من حيث الخطأ المطلق. استنادًا إلى الصيغة البارامترية للضيايبة ، يتم الاتصال بالمعادلة التفاضلية التكاملية في نظامين من النوع الثاني. تم تقديم أمثلة توضيحية لإثبات الدقة العالية والأداء الجيد للفئة الجديدة. تكشف التمثيلات الرسومية عن التناظر بين تمثيل القطع السفلية والعليا للحلول الضيايبة وقد تكون مفيدة لفهم أفضل للنموذج الغامض الاصطناعي والذكاء والعلوم الطبية والكمية.

الكلمات المفتاحية: المعادلات التكاملية التفاضلية الضبابية، السبلاين الاسي، الحل المضبوط، الحل التقريبي، المعامل الضبابي

Introduction

Fuzzy Integro-Differential Equations play very important rules in modeling dynamic systems in many applied fields such as speech processing, biological signal processing, science, electroencephalogram classification EEG, economic, engendering, communication systems and in the other sciences. In fact, most problems in nature are indistinct and uncertain, therefore the model rule are important.

Since 1972[1], both types fuzzy differential equations and integro differential equations have been studied extensively. Fuzzy derivative and its generalizations was introduced [2]. On the other hand, the fuzzy integral was introduced [3], they showed that fuzzy differential equation in the following form:

$$\begin{cases} y'(t,r) = g\big(t, y(t,r)\big) \\ \quad y(t_0, r) = y_0 \end{cases} \tag{1}$$

Has a unique solution in fuzzy case under the condition g satisfies the Lipschitz. Fuzzy Cauchy problem was studied [4]. Investigated existence and uniqueness of solutions for fuzzy Volterra integrodifferntial equations with fuzzy kernel function. [5] introduced a new class of cubic spline function approach to solve fuzzy initial value problems [6] applied the generalized spline technique and Caputo differential derivative to solve second kind of fractional integro-differential equations [7]. They Compared of the applied method with exact solutions reveals that the method was tremendously effective [8] proposed the extended trapezoidal method to solve fuzzy initial value problem that has first order.

The study of fuzzy integro differential equations was considered as a new branch of fuzzy mathematics. The analytical methods for finding the exact solutions of fuzzy integro differential equations is rather difficult [9]. So the numerical technique is the best way to resort to it. The aims of this study to improve the accuracy of the numerical solutions of fuzzy integro-differential equations. The exponential spline method has been used to solve these equations but current practice has less accuracy with error in approximating the solution for large step size. We proposed extended cubic spline technique to solve fuzzy integro-differential equations numerically. The results are expected to be more accurate as compared to be existing method [10].

The paper is organized as follows: Section 2. contains the Preliminaries. In section 3. methodology description for solving fuzzy integro-differential equations is given. In section 4, two examples are presented.The Conclusion of this paper is shown in section 5.

1. Preliminaries

In this paper,we use the following notations: $X(t_n)$ and X_n are exact solution and approximate solution respectively in time t_n.

Definition (1.1) [10]**:** A fuzzy number $\boldsymbol{\nu}$ is a fuzzy subset of a real line which it satisfies the following conditions Convexity, normality and upper semi continuous membership of bounded support.

Any fuzzy number $\boldsymbol{\nu}$ can be represented by the following parametric form $\left(\underline{\mathrm{v}}(r), \overline{\mathrm{v}}(r)\right), 0 \leq r \leq 1$. That satisfies

$\underline{\mathrm{v}}(r)$ is non-decreasing and bounded left over $0 \leq r \leq 1$

$\overline{\mathrm{v}}(r)$ is a bounded left continuous and non-increasing over $0 \leq r \leq 1$

For each r$\in [0,1]$ then $\underline{\mathrm{v}}(r) \leq \overline{\mathrm{v}}(r)$.

Definition (1.2) [11]**:** The r-level set is defend as $[u]^r = \{s; u(s) \geq r\}$, $0 \leq r \leq 1$

Consequently, $[u]^r$ can be written as close interval

$$[u]^r = [\underline{u}(r), \overline{u}(r)]$$

Definition (1.3) [12]**:** A triangular fuzzy number is a fuzzy set V in X that is characterized by a tri-ordered (a_l, a_c, a_r) in the space R^3 with $a_l \leq a_c \leq a_r$ such that $[V]^0 = [a_l, a_r]$ and $[V]^1 = \{a_c\}$. The r-level set of a triangular fuzzy number V is given by $[V]^r = [a_c - (1-r)(a_c - a_l), a_c + (1-r)(a_r - a_c)]$.

Proposition (1.4) [13]**:** Let $g: [a,b] \times [0,1] \to X$ be a fuzzy function such that $g(t,r) = \left(\underline{g}(t,r), \overline{g}(t,r)\right)$, then, If g is differentiable then $\underline{g}(t,r)$ *and* $\overline{g}(t,r)$ are differentiable functions and $g'(t,r) = \left(\underline{g}'(t,r), \overline{g}'(t,r)\right)$

Definition (1.5) [14]**:** Let $g: [a,b] \to X$. Then for any partition $\mathcal{P} = \{a = t_0, t_1, t_2, \dots, t_m = b\}$ and $\xi_i \in [t_i, t_{i+1}]$, $i = 0,1,2, \dots, m$ the definite integral of g over $a, b]$ is

$$\int_a^b g(t)dt = \lim_{\vartheta \to 0} \mathcal{M}_{\mathcal{P}}$$

Where, $\vartheta = \max\{|t_{i+1} - t_i|, i = 0,1,2, \dots, m\}$ and $\mathcal{M}_{\mathcal{P}} = \sum_{i=1}^{m} g(\xi_i)(t_{i+1} - t_i)$

When$g$ is a fuzzy and continuous function then for each fuzzy parameter $0 \leq r \leq 1$, its definite integral exists and also [7]

$$\begin{cases} \underline{\left(\int_a^b g(t,r)dt\right)} = \int_a^b \underline{g}(t,r)dt \\ \overline{\left(\int_a^b g(t,r)dt\right)} = \int_a^b \overline{g}(t,r)dt \end{cases} \quad (2)$$

Definition (1.6) [15]**:** Let $u = \left(\underline{u}(r), \overline{u}(r)\right)$ and $v = \left(\underline{v}(r), \overline{v}(r)\right)$, $0 \leq r \leq 1$ be fuzzy numbers. The distance between them is defined as follows

$$d(u,v)=\left[\int_0^1\left(\underline{u}(r)-\underline{v}(r)\right)^2 dr+\int_0^1\left(\overline{u}(r)-\overline{v}(r)\right)^2 dr\right]^{0.5} \tag{3}$$

2. Methodology Description

The fuzzy integro-differential equations

$$\begin{cases} X'(t,r)+P(t,r)\,X(t,r)=f(t,r)+\beta\int_a^b k(t,s)X(s,r)ds \\ X(a)=X_0(r) \end{cases} \tag{4}$$

Where, $\beta > 0$, k is an arbitrary given, $X'(t,r)$ is a first order derivative of the fuzzy function which defined on $[a,b]$ and is already given, r is a fuzzy parameter with values in $[0,1]$, $k(t,s)$ over $s,t \in [a,b]$ is the kernel of this equation.

In parametric form, equation (4) is represented as follows

$$\begin{cases} \underline{X'}(t,r)+\underline{P(t,r)\,X(t,r)}=\underline{f}(t,r)+\beta\int_a^b \underline{k(t,s)X(s,r)}ds \\ \overline{X'}(t,r)+\overline{P(t,r)\,X(t,r)}=\overline{f}(t,r)+\beta\int_a^b \overline{k(t,s)X(s,r)}ds \\ \underline{X}(a)=\underline{X_0}(r) \\ \overline{X}(a)=\overline{X_0}(r) \end{cases} \tag{5}$$

In addition, $\underline{P(t,r)\,X(t,r)}=\underline{P}(t,r)\underline{X}(t,r)$, $\overline{P(t,r)\,X(t,r)}=\overline{P}(t,r)\overline{X}(t,r)$, $P(t,r)=\left(\underline{P}(t,r),\overline{P}(t,r)\right)$, $\underline{k(t,s)X(s,r)}=k(t,s)\underline{X}(s,r)$, $\overline{k(t,s)X(s,r)}=k(t,s)\overline{X}(s,r)$

Suppose that the $n+1$ data points, $t_i, i=0,1,,2,,,,.n$ are the knots and increasing in order are given.Fuzzy exponential spline $S(t,r)$ through the above data points can be defined as follows

$$S(t,r)=a(r)e^{\beta(t-t_0)}+b(r)e^{2\beta(t-t_0)}+c(r)e^{3\beta(t-t_0)}+d(r)e^{4\beta(t-t_0)} \tag{6}$$

Where, $S(t,r) = \left(\underline{S}(t,r), \overline{S}(t,r)\right)$, $a(r) = \left(\underline{a}(r), \overline{a}(r)\right)$, $b(r) = \left(\underline{b}(r), \overline{b}(r)\right)$, $c(r) = \left(\underline{c}(r), \overline{c}(r)\right)$, $d(r) = \left(\underline{d}(r), \overline{d}(r)\right)$ and β is arbitrary positive real values.

By replacing t by t_0 in equation (6), we have

$$S(t_0, r) = a(r) + b(r) + c(r) + d(r) \tag{7}$$

Again, by replacing t by t_1 in equation (6), we have

$$S(t_1, r) = a(r)e^{\beta h} + b(r)e^{2\beta h} + c(r)e^{3\beta h} + d(r)e^{4\beta h} \tag{8}$$

By substituting $S(t.r)$ in the equation (6) into equation (4), we get

$$\begin{aligned} D\left(a(r)e^{\beta(t-t_0)} + b(r)e^{2\beta(t-t_0)} + c(r)e^{3\beta(t-t_0)} + d(r)e^{4\beta(t-t_0)}\right) \\ + P(t,r)\left(a(r)e^{\beta(t-t_0)} + b(r)e^{2\beta(t-t_0)} + c(r)e^{3\beta(t-t_0)} \right. \\ \left. + d(r)e^{4\beta(t-t_0)}\right) \\ = f(t,r) \\ + \beta\int_a^b k(t,s)\left(a(r)e^{\beta(s-t_0)} + b(r)e^{2\beta(s-t_0)} + c(r)e^{3\beta(s-t_0)} \right. \\ \left. + d(r)e^{4\beta(s-t_0)}\right)ds \end{aligned} \tag{9}$$

This implies

$$\begin{aligned} a(r)\left(e^{\beta(t-t_0)}(\beta + P(t,r)) - \beta\int_a^b k(t,s)e^{\beta(s-t_0)}ds\right) \\ + b(r)\left(e^{2\beta(t-t_0)}(2\beta + P(t,r)) - \beta\int_a^b k(t,s)e^{2\beta(s-t_0)}ds\right) \\ + c(r)\left(e^{3\beta(t-t_0)}(3\beta + P(t,r)) - \beta\int_a^b k(t,s)e^{3\beta(s-t_0)}ds\right) \\ + d(r)\left(e^{4\beta(t-t_0)}(4\beta + P(t,r)) - \beta\int_a^b k(t,s)e^{4\beta(s-t_0)}ds\right) \\ = f(t,r) \end{aligned} \tag{10}$$

Now, we use the following notations

$$M_1(t) = e^{\beta(t-t_0)}\big(\beta + P(t,r)\big) - \beta\int_a^b k(t,s)e^{\beta(s-t_0)}ds \tag{11}$$

$$M_2(t) = e^{2\beta(t-t_0)}\big(2\beta + P(t,r)\big) - \beta\int_a^b k(t,s)e^{2\beta(s-t_0)}ds \tag{12}$$

$$M_3(t) = e^{3\beta(t-t_0)}\big(3\beta + P(t,r)\big) - \beta\int_a^b k(t,s)e^{3\beta(s-t_0)}ds \tag{13}$$

$$M_4(t) = e^{4\beta(t-t_0)}\big(4\beta + P(t,r)\big) - \beta\int_a^b k(t,s)e^{4\beta(s-t_0)}ds \tag{14}$$

Consequently,

$$\mathcal{M} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ M_1(t_1) & M_2(t_1) & M_3(t_1) & M_4(t_1) \\ \vdots & \vdots & \vdots & \vdots \\ M_1(t_m) & M_2(t_m) & M_3(t_m) & M_4(t_m) \end{bmatrix} \tag{15}$$

$$\mho(r) = \begin{bmatrix} a(r) \\ b(r) \\ c(r) \\ d(r) \end{bmatrix} \tag{16}$$

$$E(r) = \begin{bmatrix} X_0(r) \\ f(t_1,r) \\ \vdots \\ f(t_n,r) \end{bmatrix} \tag{17}$$

For each r, $\mathcal{M}$ and $E(r)$ are $(n+1)\times 4$ and $(n+1)\times 1$ matrices respectevely, $\mho(r)$ is unknown vector.

If n$\geq$ 3 *then our system* have $n+1$ equations and 4 coefficients therefore,

$$\mathcal{M}^{\tau}\mathcal{M}\mho(r) = \mathcal{M}^{\tau}E(r) \tag{18}$$

where, $\mathcal{M}^{\tau}$ is transpose matrix of $\mathcal{M}$.

3. Illustrative examples

To show the efficiency and accuracy of the propose technique with various values of step size, we consider the following two examples.

Example (3.1): Consider the following integro-differential equations taken from (4)

$$\begin{cases} X'(t,r) + X(t,r) = \big((3+3r)\sinh(t), (8-2r)\sinh(t)\big) + \int_0^1 (t-s)X(s,r)ds \\ X(0,r) = \big((3+3r),(8-2r)\big) \ , \ t \in [0,1], 0 \le r \le 1 \end{cases} \tag{19}$$

The exact solution is given by

$$X(t,r) = \big((3+3r)\cosh(t), (8-2r)\cosh(t)\big) \tag{20}$$

To compare we use the formula $d(X_n, X(t_n)) = \sup_{0\le r\le 1} \max(\underline{X_n} - \underline{X(t_n)}, \overline{X_n} - \overline{X(t_n)})$

Let us compute the approximate solution of equation (19) by using exponential spline method. Here, we take step size $h = 0.1,\ h = 0.01\ and\ \ h = 0.001$

Consider equation (19), then

$$P(t,r) = 1\ , f(t,r) = \big((3+3r)\sinh(t), (8-2r)\sinh(t)\big)\ , a = 0, b = 1\ and\ \ k(t,s) = (t-s)$$

Approximate solutions $\underline{X_n}$, $\overline{X_n}$ can be found by solving equations in (18) (see Fig. 1, 2, 3) And Table 1, 2, 3,4)

Table 1: The fuzzy coefficients of equation (6) are computed when $h = 0.01, t = 0.3, \beta = 1$

r	$\underline{a}(r)$	$\overline{a}(r)$	$\underline{b}(r)$	$\overline{b}(r)$	$\underline{c}(r)$	$\overline{c}(r)$	$\underline{d}(r)$	$\overline{d}(r)$
0	6.9681	18.5816	-6.2435	-16.6493	2.5832	6.8886	-0.3684	-0.9824
0.1	7.6649	18.1171	-6.8678	-16.2331	2.8415	6.7164	-0.4052	-0.9578
0.2	8.3617	17.6525	-7.4922	-15.8168	3.0999	6.5442	-0.4421	-0.9332
0.3	9.0585	17.1880	-8.1165	-15.4006	3.3582	6.3720	-0.4789	-0.9087
0.4	9.7553	16.7234	-8.7409	-14.9844	3.6165	6.1997	-0.5157	-0.8841
0.5	10.4521	14.5681	-9.3652	-16.2589	3.8748	6.0275	-0.5526	-0.8596
0.6	11.1490	15.7944	-9.9896	-14.1519	4.1332	5.8553	-0.5894	-0.8350
0.7	11.8458	15.3298	-10.6139	-13.7357	4.3915	5.6831	-0.6262	-0.8104
0.8	12.5426	14.8653	-11.2383	-13.3194	4.6498	5.5109	-0.6631	-0.7859
0.9	13.2394	14.4007	-11.8626	-12.9032	4.9081	5.3387	-0.6999	-0.7613
1	13.9362	13.9362	-12.4870	-12.4870	5.1665	5.1665	-0.7368	-0.7368

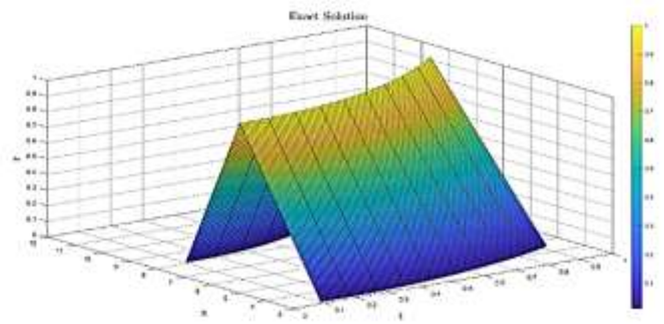

Figure 1: Exact Solution

Table 2: $h = 0.1,\ \beta = 1,$

t	D
0	0
0.3	0.0057
0.5	0.0013
0.7	0.0030
0.9	0.0078

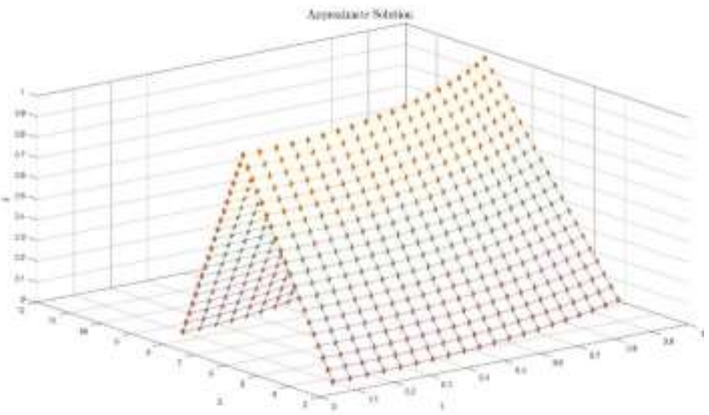

Figure 2: Approximate Solution

Table 3: $h = 0.01, \quad \beta = 1$

t	D
0	0
0.3	0.0040
0.5	0.0008
0.7	0.0015
0.9	0.0039

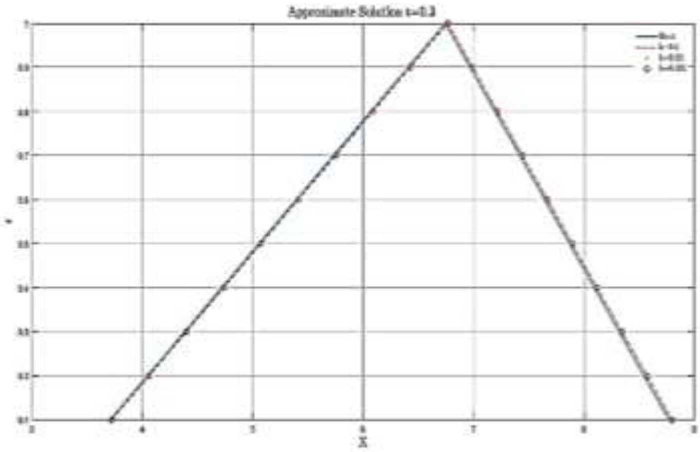

Figure 3: Exact and Approximate Solution at t=0.3

Table 4: $h = 0.001, \ \beta = 1$

t	D
0	0
0.3	3.7×10^{-5}
0.5	7.58×10^{-5}
0.7	0.0013
0.9	0.0036

Example (3.2): Consider the following integro-differential equations

$$\begin{cases} X'(t,r) + 2\,X(t,r) = \big((1+r)(1+t),(3-r)(1+t)\big) - \int_0^2 X(s,r)ds \\ X(0,r) = (1+r,3-r) \ \ , \ \ t \in [0,2], 0 \le r \le 1 \end{cases} \tag{21}$$

$$P(t,r) = 2, f(t,r) = \big((1+r)(1+t),(3-r)(1+t)\big), \quad \beta = -1, \quad k(t,s) = 1$$

The exact solution is given by

$$X(t,r) = \big((1-e^{-t})(1+r) + e^{-t}(1-t)(1+r), (1-e^{-t})(3-r) + e^{-t}(1-t)(3-r)\,\big) \tag{22}$$

Approximate solutions $\underline{X_n}$, $\overline{X_n}$ can be found by solving equations in (22) (see Figs. 4, 5. ,6.) and Tables(5,6,7)

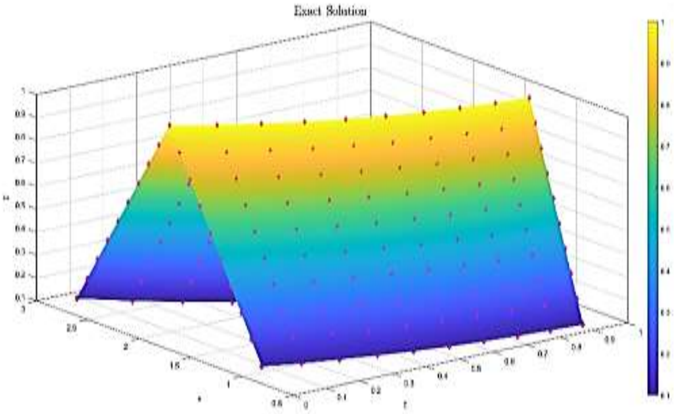

Figure 4: Exact Solution

Table 5: $h = 0.01$,

t	D
0	0
0.3	0.0040
0.6	0.0037
0.9	0.0052
1.2	0.0114
1.5	0.0652
1.8	0.0625

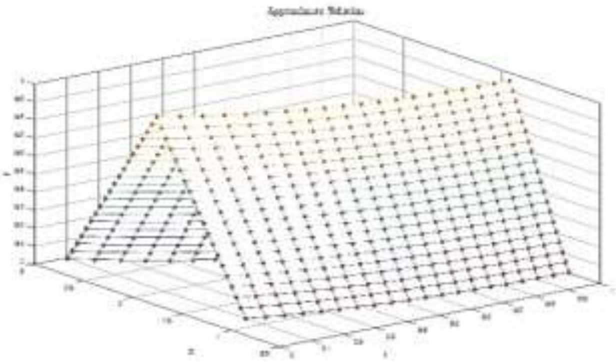

Figure 5: Approximate Solution

Table 6: $h = 0.01$

t	D
0	0
0.3	0.0023
0.6	0.0883
0.9	0.0466
1.2	0.0117
1.5	0.0448
1.8	0.0603

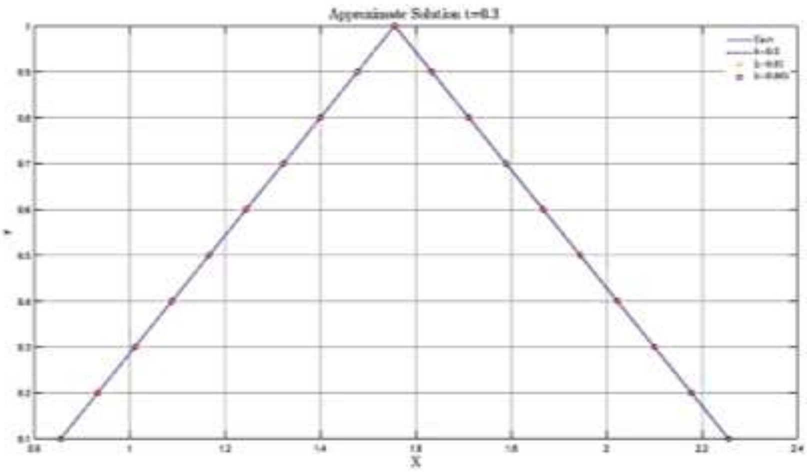

Figure 6: Exact and Approximate Solution at t=0.3

Table 7: $h =$ 0.01

t	d
0	0
0.3	0.0026
0.6	0.0858
0.9	0.0449
1.2	0.0120
1.5	0.0426
1,8	0.0604

Conclusion

In this work, a new class of exponential spline function method is introduced for solving fuzzy integro-differential equations subject to fuzzy initial conditions. This technique proves its efficiency and reliability in solving of these equations by providing the best approximate solutions. The numerical outputs obtained using the proposed techniques are comparable to the exact solutions of our proposed model. We showed that the step size h played a fundamental and important role in reducing the error rate which resulting from the approximation of solutions for Fuzzy Integro-Differential Equations. Thus, our work in this paper, one can extend this method to solve fractional-order fuzzy initial value problems. Finally, we would like to refer that the proposed equation can be applied to various fields such as environmental, medicine, economy, engineering and biomedical.

References

1. C. E. Shanon, A mathematical Theory of Communication, Bell systems technology, 27, 379-423, 623-656(1948)
2. S. Seikkala, On the fuzzy initial value problem. Fuzzy sets and systems, 24(3), 319-330(1987)
3. D. Dubois, H. Prade, Towards fuzzy differential calculus part 1: Integration of fuzzy mappings. Fuzzy sets and Systems, 8(1), 1-17(1982)
4. O. Kaleva, The Cauchy problem for fuzzy differential equations. Fuzzy sets and systems, 35(3), 389-396(1990)
5. S. Hajighasemi, T. Allahviranloo, M. Khezerloo, M. Khorasany, S. Salahshour, Existence and uniqueness of solutions of fuzzy Volterra integro-differential equations, In: Information Processing and Management of Uncertainty in Knowledge-Based Systems. Applications: 13th International Conference, IPMU 2010, Dortmund, Germany, June 28–July 2, Proceedings, Part II 13 pp, 491-500, Springer Berlin Heidelberg(2010)

6. F. Ishak, N. Chaini, Numerical computation for solving fuzzy differential equations, Indonesian Journal of Electrical Engineering and Computer Science, 16(2), 1026-1033(2019)
7. S. S. Chang, L. A. Zadeh, On fuzzy mapping and control. IEEE Transactions on Systems, Man, and Cybernetics, (1), 30-34(1972)
8. R. Jr Goetschel, W. Voxman, Fuzzy circuits. Fuzzy Sets and Systems, 32(1), 35-43(1989)
9. Y. Chalco-Cano, H. Roman-Flores, on new solutions of fuzzy differential equations, Chaos, Solitons & Fractals, 38(1), 112-119(2008)
10. R. Bello, R. Falcon, J. L. Verdegay, (Eds.). Uncertainty management with fuzzy and rough sets, (Recent advances and applications, 2019), 127- 139
11. D. Dubois, H. Prade, (Eds.)., Fundamentals of fuzzy sets, Vol. 7, (Springer Science & Business Media, 2012)
12. P. Darabi, S. Moloudzadeh, H. Khandani, A numerical method for solving first-order fully fuzzy differential equation under strongly generalized H-differentiability, Soft Computing, 20, 4085-4098 (2016)
13. S. Karpagappriya, N. Alessa, P. Jayaraman, K. Loganathan, A Novel Approach for Solving Fuzzy Differential Equations Using Cubic Spline Method, Mathematical Problems in Engineering, 1-9(2021)
14. N. N. Hasan, D. A. Hussien, Generalized Spline method for integro-differential equations of fractional order, Iraqi Journal of Science, 1093-1099(2018)
15. M. Zeinali, Approximate solution of fuzzy Hammerstein integral equation by using fuzzy B-spline series, Sohag Journal of Mathematics, 4, 19-25(2017)