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Numerical Approximate Solution of Fuzzy Volterra Nonlinear Integro Differential Equation

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ABSTRACT

In this work, approximate solutions to fuzzy integro-differential equations refer to numerical methods or techniques used to obtain approximate solutions to differential equations involving fuzzy sets and integro-differential operators. Fuzzy integro-differential equations have recently increased as a model for many problems in the fields of science and technology and so we propose efficient approximate methods to solve nonlinear fuzzy integro-differential equations. Therefore, nonlinear fuzzy integro-differential equations are usually complex to solve analytically, and exact solutions are scarce. We combined two numerical methods to obtain an approximate solution to the nonlinear fuzzy Volterra integro-differential equation of the first and second order. Specifically, we combined the fuzzy Euler predictor-corrector and fuzzy Taylor expansion with fuzzy Newton-Cotes integration, respectively. Finally, the applicability and validity of the numerical approaches are demonstrated, and the findings show that the proposed methods' convergence and accuracy agree closely with the exact solution. These findings are shown in tables and graphical figures.

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1. INTRODUCTION

Recent years have seen an increase in the study of fuzzy integro-differential equations, which are crucial mathematical models for numerous issues in a variety of disciplines, including physics, chemistry, biology, engineering, etc. This growth in the field of fuzzy control system research was reflected in a fast increase in interest in this subject. In actuality, the FIDE's evolutionary trend pushed scientists to develop accurate and efficient solutions. As a result, it has been emphasized in numerous past research studies in the literature, for example, the existence of solution and its uniqueness has been studied for fuzzy Volterra integro-differential equations (FVIDE) in [1], apply of Laplace transformation with the Adomian decomposition method in [2], The fuzzy general linear method is investigated in [3], the extended difference Euler technique is applied in [4], and the Fuzzy differential transform method is used [5], Legendre wavelet method is applied in [6], He's homotopy perturbation method is used [7], different numerical methods are proposed in [8], the expansion method is discussed in [9], the approximate solution is found by the extended form of homotopy perturbation method [10], the Sumudu decomposition method was used in [11], the residual power series method is presented in [12].

In this paper, we introduce two numerical methods by combining each of the fuzzy Euler predictor-corrector [13] and the fuzzy Taylor expansion [14] with fuzzy Newton-Cotes integration [15], respectively, to solve the fuzzy nonlinear Volterra integro differential equation (FNLVIDE) of the first and second order. The remainder of this paper is organized as follows. Section 2 introduces the fuzzy calculus through some elementary and necessary preliminaries. Section 3 introduces the FNLVIDE, and the proposed numerical method is presented in Section 4. Practical implementation of the proposed method is introduced in Section 5. In Section 6, examples of how to use the suggested method are provided. Finally, Section 7 displayed the work's conclusions.

1. PRELIMINARIES

This section provides basic definitions used throughout the paper. Let y be a Fuzzy number satisfying the following:

Definition .1 [14] A fuzzy number is a set $y : \mathcal{R} \to I = [0,1]$That satisfies the following:
i. y is upper semi-continuous and fuzzy convex.
ii. Normal and closure $(supp\ y)$ is compact, where $supp\ y = \{\omega \in \mathcal{R} : y(\omega) > 0\}$represent the support of y.
Let E Be the set of all fuzzy numbers on $\mathcal{R}$. The $\rho - level$Set of y, is denoted by

$$[y]^{\rho} = \begin{cases} \{\omega \in R : y(\omega) > 0\} & if\ 0 < \rho \le 1 \\ cl(supp\ y) & if\ \rho = 0 \end{cases}$$

where $\underline{y}(\rho)$Is the left-hand endpoint and $\overline{y}(\rho)$ is the right-hand endpoint of $[y]^{\rho}$respectively and $y(\rho) = [\underline{y}(\rho), \overline{y}(\rho)]$ As a closed bounded interval.

Definition .2 [15] A fuzzy number y is a pair $[\underline{y}(\rho), \overline{y}(\rho)]$, $0 \leqslant \rho \leqslant 1$, which satisfy the following properties:

i. $\overline{v}(\omega) and\ \underline{y}(\rho)$ are bounded monotonic non-increasing and non-decreasing left continuous functions, respectively.

ii. $\underline{y}(\rho) \leqslant \overline{v}(\omega), 0 \leqslant \omega \leqslant 1$

Let $y(\rho) = [\underline{y}(\rho), \overline{y}(\rho)]$ & $z(\omega) = [\underline{z}(\rho), \overline{z}(\rho)]$ are different arbitrary fuzzy numbers, and $k > 0$, we define the operations; addition $(y + z)$, subtraction $(y - z)$ and multiplication by scaler k as follow,

$(y+z)$: $(\underline{y+z})(\rho) = \underline{y}(\rho) + \underline{z}(\rho)$, $(\overline{y+z})(\rho) = \overline{y}(\rho) + \overline{z}(\rho)$

$(y-z)$: $\left(\underline{y-z}\right)(\rho) = \underline{y}(\rho) - \underline{z}(\rho)$, $(\overline{y-z})(\rho) = \overline{y}(\rho) - \overline{z}(\rho)$

ky: $(\underline{ky})(\rho) = k\underline{y}(\rho)$, $(\underline{ky})(\rho) = k\underline{y}(\rho)\ if\ k \geqslant 0$

$(\overline{ky})(\rho) = k\overline{y}(\rho)$, $(\overline{ky})(\rho) = k\overline{y}(\rho)\ if\ k < 0$

Definition .3 [16] For any two fuzzy numbers y and z, the distance between them given by $\mathcal{D}: E \times E \to \mathcal{R}^+ \cup \{0\}$, as:

$$\mathcal{D}(y,z) = sup_{\rho \in [0,1]} max\{|\underline{y}(\rho) - \underline{z}(\rho)|, |\overline{y}(\rho) - \overline{z}(\rho)|\}$$

where sup represents the supremum operator, $\mathcal{D}$ is a metric in E and has the next properties:
i. $\mathcal{D}(y + w, z + w) = \mathcal{D}(v, z)\ for\ all\ y, z, w \in E;$
ii. $\mathcal{D}(k.y, k.z) = |k|\mathcal{D}(y, z)\ for\ all\ k \in \mathcal{R}\ and\ y, z \in E;$
iii. $\mathcal{D}(y + w, z + u) \leqslant \mathcal{D}(y, z) + \mathcal{D}(w, u) for\ all\ y, z, w, u \in E;$
iv. $(\mathcal{D}, E)\ is\ a\ complete\ metric\ space.$

Definition .4 [17] Let $f : [a, b] \to \mathcal{R}$ is a fuzzy function, for any fixed $\omega_0 \in \mathcal{R}$ and $\delta > 0$ there exists $\delta > 0$ such that if $|\omega - \omega_0| < \delta$ then $\mathcal{D}(f(\omega) - f(\omega_0)) < \varepsilon$, and f is continuous.

Definition .5 [18] Let $y, z \in E$, if there exists $w \in E$ such that $y = z + w$, then w is said the H-difference of y and z,and denoted as $y \ominus z$.

Definition .6 [19] Let I be a real interval. A mapping $y : I \to E$ is called a fuzzy process, and we denote the $r - level$ set by $y_{\rho}(\omega) = [\underline{y}(\omega, \rho), \overline{y}(\omega, \rho)]$, The derivative $\widetilde{y}'(\omega)$of $\widetilde{y}$ is defined by:

$$y'_{\rho}(\omega) = [\underline{y'}(\omega, \rho), \overline{y'}(\omega, \rho)]$$

provided that is an equation defines a fuzzy number $\widetilde{y'}(\omega) \in E(\omega)$.

Definition.7[19] The fuzzy integral of a fuzzy process $\tilde{y}$, $\int_a^b y(\omega)dt$ for

$$\int_a^b y_{\rho}(\omega) = [\int_a^b \underline{y}(\omega, \rho)dt, \int_a^b \overline{y}(\omega, \rho)dt]$$

provided that the Lebesgue integrals on the right exist.

Definition .8 [20] A function $f: (a, b) \to E_1$ is called H-differentiable at $\hat{x} \in (a, b)$ if, for $h > 0$ sufficiently small, there exist the H-differences $f(\hat{x} + h) - f(\hat{x}), f(\hat{x}) - f(\hat{x} - h)$, and an element $f'(\hat{x}) \in E^1$ such that:

$$\lim_{h \to 0^+} D(\frac{f(\hat{x} + h) - f(\hat{x})}{h}, f'(\hat{x})) = \lim_{h \to 0^-} D(\frac{f(\hat{x}) - f(\hat{x} - h)}{h}, f'(\hat{x})) = 0$$

Then $f'(\hat{x})$ is called the fuzzy derivative of f at $\hat{x}$.

Definition .9 [21] A function f, defined on [*a*, *b*], is said to satisfy a Lipschitz condition on [*a*, *b*] if there exists a constant *L* > 0 such that

$$d_{\infty}(f(\omega, x), f(\omega, y)) \leqslant L d_{\infty}(x, y)$$

2. FUZZY NONLINEAR VOLTERRA INTEGRO DIFFERENTIAL EQUATION (FVIDE)

In this section, We talk about the FNVIDE, and we use uncertain primal conditions. Consider the following:

$$y^{(n)}(\omega) = f(\omega) + \int_a^\omega k(\omega, s)F(y(s))ds\,, \quad a \le s, \omega \le b \tag{1}$$

with initial conditions: $y^i(\omega) = y^i,\ i = 0, \dots, n-1$.
where $n = 1, 2$ refers to the order of the FIDE, the known functions f and k are continuous and defined on $[a, b]$ and $[a, b] \times \mathcal{R}$, respectively, and the function $y(\omega)$Is an unknown. Note that we conduct an analytical investigation for FVIDE in this part. The given Eq. (1), by the fuzzy concept, becomes:

$$y^{(n)}(\omega, \rho) = f(\omega) + \int_a^\omega k(\omega, \rho)F(y(s, \rho))ds\,, \quad a \le s, \omega \le b \tag{2}$$

with initial conditions: $y^i(\omega, r) = y^i,\ i = 0, \dots, n-1$.
Now, the parametric form of FNLVIDE based to definition 2 has been introduced. Let. $(\underline{y^n}(\omega, r), \overline{y^n}\,(\omega, r))$, $(\underline{f}\,(\omega, \rho), \overline{f}\,(\omega, \rho))$ and $(\underline{F}(y(s, \rho)), \overline{F}(y(s, \rho))), 0 \leqslant \rho \leqslant 1$ and $a \leqslant \omega \leqslant b$, be parametric forms of $y^n(\omega), f(\omega)$ and $y(s)$, respectively; the parametric form of FVIDE looks like this:

$$\begin{aligned} \underline{y^{(n)}}(\omega) &= \underline{f}(\omega) + \int_a^\omega k(\omega, s)\underline{F}(y(s, \rho))ds\,, \quad a \le s, \omega \le b \\ \overline{y^{(n)}}(\omega) &= \overline{f}(\omega) + \int_a^\omega k(\omega, s)\overline{F}(y(s, \rho))ds\,, \quad a \le s, \omega \le b \end{aligned} \tag{3}$$

with initial conditions: $\underline{y^i}(\omega, r) = \underline{y_i}\,,\ \overline{y^i}(\omega, \rho) = \overline{y_i}\,,\ i = 0, \dots, n-1$
We can observe that Eq. (3) is a system of nonlinear Volterra integro-differential equations in the crisp case for each $0 \leqslant \rho \leqslant 1$and $\omega \in [a, b]$.

3. PROPOSED NUMERICAL METHOD

In this section, we will first present the approach to address the fuzzy derivative, which, represented by predictor-corrector backward then, Introduce the approach to address the fuzzy integral, which is represented by a fuzzy quadrature method.

3.1 Numerical Fuzzy Derivatives

Firstly, consider the first-order fuzzy differential equation. $y' = f\,(\omega, y)$, where y is a fuzzy function of ω and $f\,(\omega, y)$ is a fuzzy function of a crisp variable t and a fuzzy variable y and y' is a fuzzy derivative of y. If an initial value is given, a fuzzy problem of first order is:

$$\begin{aligned} y' &= f(\omega, y), \quad \omega \ge \omega_0 \\ y(\omega_0) &= y_0 \end{aligned} \tag{4}$$

Sufficient conditions for the existence of a unique solution to Eq.(4.1) are:

- f continuous,
- Lipschitz condition $d_\infty(\,f(\omega, x), f(\omega, y)) \leqslant L d_\infty(x, y), L > 0$.

Single-step and multi-step methods to solve fuzzy problems are as follows, where h is called the step size, that is, the length of the interval over which the approximation is made.

- Euler method:

$$\begin{aligned} \underline{y}_{i+1}(\omega, \rho) &= \underline{y}_i(\omega, \rho) + h.\underline{y'}_i(\omega, \rho), \\ \overline{y}_{i+1}(\omega, \rho) &= \overline{y}_i(\omega, \rho) + h.\overline{y'}_i(\omega, \rho),\ i = 0,1,2, \dots. \end{aligned} \tag{5}$$

- Euler predictor-corrector method:

$$\begin{aligned} \underline{y}_{i+1}(\omega, \rho) &= \underline{y}_i(\omega, \rho) + \tfrac{h}{2}.\left[\underline{y'}_i(\omega, \rho) + \underline{y'}_{i+1}(\omega, \rho)\right], \\ \overline{y}_{i+1}(\omega, \rho) &= \overline{y}_i(\omega, \rho) + \tfrac{h}{2}.\left[\overline{y'}_i(\omega, \rho) + \overline{y'}_{i+1}(\omega, \rho)\right], i = 0,1,2, \dots. \end{aligned} \tag{6}$$

We get

$$\begin{aligned} \underline{y'}_{i+1}(\omega, \rho) &= \tfrac{2}{h}\left[\underline{y}_{i+1}(\omega, \rho) - \underline{y}_i(\omega, \rho)\right] - \underline{y'}_i(\omega, \rho), \\ \overline{y'}_{i+1}(\omega, \rho) &= \tfrac{2}{h}\left[\overline{y}_{i+1}(\omega, \rho) - \overline{y'}_{i+1}(\omega, \rho)\right] - \overline{y}_i(\omega, \rho)\,,\ i = 0,1,2, \dots. \end{aligned} \tag{7}$$

Secondly, consider the second-order fuzzy differential equation as

$$\begin{aligned} y'' &= f(\omega, y), \quad \omega \in [\omega_0, T\,] \\ y'(\omega_0) &= y'_0\,, \quad y(\omega_0) = y_0 \end{aligned} \tag{8}$$

With the same sufficient conditions to demonstrate the existence of a unique solution of Eq. (4), by including terms from the fuzzy Taylor series expansion, a high-accuracy divided-difference formula can be produced. For example, the forward Taylor series expansion can be expressed as:

$$\underline{y}_{i+1}(\omega,\rho) = \underline{y}_i(\omega,\rho) + h.\underline{y'}_i(\omega,\rho) + \frac{h^2}{2}.\underline{y''}(\omega,\rho) + \cdots,$$
$$\overline{y}_{i+1}(\omega,\rho) = \overline{y}_i(\omega,\rho) + h.\overline{y'}_{i+1}(\omega,\rho) + \frac{h^2}{2}\overline{y''}(\omega,\rho) + \cdots,\ i = 0,1,2,.... \quad (9)$$

We now retain the second-derivative term by substituting the following approximation of the second derivative in Eq. (8)

$$\underline{y''}_i(\omega,\rho) = \frac{2}{h^2}[\underline{y}_{i+1}(\omega,\rho) - \underline{y}_i(\omega,\rho) - h.\underline{y'}_i(\omega,\rho)],$$
$$\overline{y''}_{i+1}(\omega,\rho) = \frac{2}{h^2}[\overline{y}_{i+1}(\omega,\rho) - \overline{y}_i(\omega,\rho) - h.\overline{y'}_i(\omega,\rho),\ i = 0,1,2,.... \quad (10)$$

3.2 Numerical Fuzzy Integration

Numerical methods for the integration of fuzzy functions are considered the Fuzzy Trapezoidal method and Simpson method is calculated by integration of fuzzy functions.

• Fuzzy trapezoidal method:

$$\int_{\omega_0}^{\omega_n} \underline{y}(\omega,\rho) = h[\frac{1}{2}\underline{y}(\omega_0,\rho) + \sum_{i=1}^{n-1}\underline{y}(\omega_i,\rho) + \frac{1}{2}\underline{y}(\omega_n,\rho)]$$
$$\int_{\omega_0}^{\omega_n} \overline{y}(\omega,\rho) = h[\frac{1}{2}\overline{y}(\omega_0,\rho) + \sum_{i=1}^{n-1}\overline{y}(\omega_i,\rho) + \frac{1}{2}\overline{y}(\omega_n,\rho)] \quad (11)$$

• Fuzzy Simpson method:

$$\int_{\omega_0}^{\omega_n} \underline{y}(\omega,\rho) = h[\frac{1}{3}\underline{y}(\omega_0,\rho) + \frac{4}{3}\sum_{i=1}^{n-1}\underline{y}(\omega_i,\rho) + \frac{1}{3}\underline{y}(\omega_n,\rho)]$$
$$\int_{\omega_0}^{\omega_n} \overline{y}(\omega,\rho) = h[\frac{1}{3}\overline{y}(\omega_0,\rho) + \frac{4}{3}\sum_{i=1}^{n-1}\overline{y}(\omega_i,\rho) + \frac{1}{3}\overline{y}(\omega_n,\rho)] \quad (12)$$

4. IMPLEMENTATIONS OF THE PROPOSED METHOD

The type of equation and methods of solution have been presented in previous sections, as a framework for our work. To solve the FNLVIDE, the implementation problem of the suggested framework will be discussed in this part. At the start, we use a new formula to determine the numerical solution for the first order in Equation (3), which as:

$$\underline{y'}(\omega,\rho) = \underline{f}(\omega,\rho) + \int_a^{\omega} k(\omega,s)\underline{F}(y(s,\rho))ds$$
$$\overline{y'}(\omega,\rho) = \overline{f}(\omega,\rho) + \int_a^{\omega} k(\omega,s)\overline{F}(y(s,\rho))ds \quad (13)$$

with initial conditions $\underline{y}(\omega,\rho) = \underline{y_0},\ \overline{y}(\omega_0,\rho) = \overline{y_0}$, the integral vanishes for $\omega = \omega_0$ and first order in Eq. (13) will become:

$$\underline{y'_0}(\omega,\rho) = \underline{f_0}(\omega,\rho)$$
$$\overline{y'_0}(\omega,\rho) = \overline{f_0}(\omega,\rho) \quad (14)$$

The first derivative and integral in Eq. (13) can be treated by using Eq. (7) and Eq. (11), yields

$$\underline{y}_{i+1}(\omega,\rho) = \frac{h}{2}[\underline{y_i}'(\omega,\rho) + \frac{2}{h}\underline{y}_i(\omega,\rho) + \underline{f}(\omega,\rho) + h.k(\omega,s)(\frac{1}{2}\underline{F}(y(a,\rho)) + \sum_{j=1}^{n-1}\underline{F}(y(\omega_j,\rho)) + \frac{1}{2}\underline{F}(y(\omega,\rho)))],$$
$$\overline{y}_{i+1}(\omega,\rho) = \frac{h}{2}[\overline{y_i}'(\omega,\rho) + \frac{2}{h}\overline{y_i}(\omega,\rho) + \overline{f}(\omega,\rho) + h.k(\omega,s)(\frac{1}{2}\underline{F}(y(a,\rho)) + \sum_{j=1}^{n-1}\overline{F}(y(\omega_j,\rho)) + \frac{1}{2}\overline{F}(y(\omega,\rho)))] \quad (15)$$

As well, we solve Eq. (13) using Eq. (7) and Eq. (12), get

$$\underline{y}_{i+1}(\omega,\rho) = \frac{h}{2}[\underline{y'}(\omega,\rho) + \frac{2}{h}\underline{y}_i(\omega,\rho) + \underline{f}(\omega,\rho) + h.k(\omega,s)(\frac{1}{3}\underline{F}(y(a,\rho)) + \frac{4}{3}\sum_{j=1}^{n-1}\underline{F}(y(\omega_i,\rho)) + \frac{1}{3}\underline{F}(y(\omega,\rho)))]$$
$$\overline{y}_{i+1}(\omega,\rho) = \frac{h}{2}[\overline{y'}(\omega,\rho) + \frac{2}{h}\overline{y_i}(\omega,\rho) + \overline{f}(\omega,\rho) + h.k(\omega,s)(\frac{1}{3}\overline{F}(y(a,\rho)) + \frac{4}{3}\sum_{j=1}^{n-1}\overline{F}(y(\omega_i,\rho)) + \frac{1}{3}\overline{F}(y(\omega,\rho)))] \quad (16)$$

Now, we drive a formula to find numerical solution for second order in Eq. (3), which as:

$$\underline{y''}(\omega,\rho) = \underline{f}(\omega,\rho) + \int_a^{\omega} k(\omega,s)\underline{F}(y(s,\rho))ds$$
$$\overline{y''}(\omega,\rho) = \overline{f}(\omega,\rho) + \int_a^{\omega} k(\omega,s)\overline{F}(y(s,\rho))ds \quad (17)$$

with initial conditions $\underline{y'}(\omega_0,\rho) = \underline{y'_0},\ \overline{y'}(\omega_0,\rho) = \overline{y'_0}$ and $\underline{y}(\omega_0,\rho) = \underline{y_0},\ \overline{y}(\omega_0,\rho) = \overline{y_0}$, the integral vanishes for $\omega = \omega_0$ and first order in Eq.(13) will become:

$$\underline{y_0}''(\omega,\rho) = \underline{f_0}(\omega,\rho)$$
$$\overline{y_0}''(\omega,\rho) = \overline{f_0}(\omega,\rho) \tag{18}$$

so by using Eq. (10) and Eq. (11), the approximation solution obtained for Eq. (17) as

$$\underline{y}_{i+1}(\omega,\rho) = \frac{h^2}{2}[\frac{2}{h}\underline{y}'(\omega,\rho) + \frac{2}{h^2}\underline{y}_i(\omega,\rho) + \underline{f}(\omega,\rho) + h.k(\omega,s)(\frac{1}{3}\underline{F}(y(a,\rho)) + + \sum_{j=1}^{n-1}\underline{F}(y(\omega_i,\rho)) + \frac{1}{2}\underline{F}(y(\omega,\rho)))$$
$$\overline{y}_{i+1}(\omega,\rho) = \frac{h^2}{2}[\frac{2}{h}\overline{y'}(\omega,\rho) + \frac{2}{h^2}\overline{y_i}(\omega,\rho) + \overline{f}(t,\rho) + h.k(\omega,s)(\frac{1}{3}\overline{F}(y(a,\rho)) + \sum_{j=1}^{n-1}\overline{F}(y(\omega_i,\rho)) + \frac{1}{2}\overline{F}(y(\omega,\rho)))] \tag{19}$$

Substitute Eq. (10) and Eq. (12) into Eq. (17) and we conclude that

$$\underline{y}_{i+1}(\omega,\rho) = \frac{h^2}{2}[\frac{2}{h}\underline{y}'(\omega,\rho) + \frac{2}{h^2}\underline{y}_i(\omega,\rho) + \underline{f}(\omega,\rho) + h.k(\omega,s)(\frac{1}{3}\underline{F}(y(a,\rho)) + \frac{4}{3}\sum_{j=1}^{n-1}\underline{F}(y(\omega_j,\rho)) + \frac{1}{3}\underline{F}(y(\omega,\rho)))]$$
$$\overline{y}_{i+1}(\omega,\rho) = \frac{h^2}{2}[\frac{2}{h}\overline{y'}(\omega,\rho) + \frac{2}{h^2}\overline{y_i}(\omega,\rho) + \overline{f}(t,\rho) + h.k(\omega,s)(\frac{1}{3}\overline{F}(y(a,\rho)) + \frac{4}{3}\sum_{j=1}^{n-1}\overline{F}(y(\omega_j,\rho)) + \frac{1}{3}\overline{F}(y(\omega,\rho)))] \tag{20}$$

5. ILLUSTRATIVE EXAMPLES

In this section, we will demonstrate the usefulness of the suggested approach by evaluate its performance through the extraction of solutions for fuzzy volterra integro-differential equations. To this end, the resulting examples will be offered.

EXAMPLE 1. Consider the following first order fuzzy nonlinear Volterra integro-differential equation:

$$\underline{y}'(\omega,\rho) = \rho - \frac{\rho^2\omega^5}{10} + \int_0^{\omega}\frac{s^2}{2}\underline{y}^2(s,\rho)ds,$$
$$\overline{y'}(\omega,\rho) = (2-\rho) - \frac{(2-\rho)^2\omega^5}{10} + \int_0^{\omega}\frac{s^2}{2}\overline{y^2}(s,\rho)ds \tag{21}$$

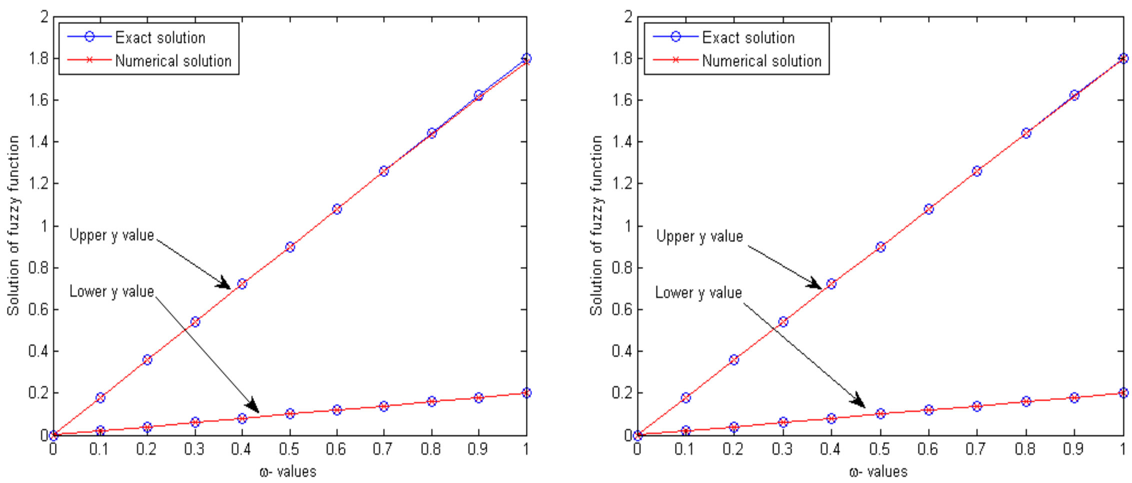
with initial condition $y(0,\rho) = 0$, $0 \leqslant \omega$, $s \leqslant 1$ and $0 \leqslant \rho \leqslant 1$. The exact solution of the given fuzzy integro-differential solution is $y(\omega,\rho) = (\rho\ , 2-\rho)$, $0 \leqslant \omega$, $s \leqslant 1$. The numerical results and their graphical visualization are displayed in Tables I and II and Fig. I, respectively.

Table 1: Euler predictor-corrector with fuzzy Trapezoidal method, for $\rho = 0.2$ for example 1.

ω	$\underline{y}$	Exact Solution of $\underline{y}$	Least Squares error	$\overline{y}$	Exact Solution of $\overline{y}$	Least Squares error
0	0.0000	0.0000	0.0000	0.0000	0.000	0.0000
0.1	0.0200	0.0200	0.0000	0.1800	0.1800	0.0000
0.2	0.0400	0.0400	0.0000	0.3600	0.3600	0.0000
0.3	0.0600	0.0600	0.0000	0.5400	0.5400	0.0000
0.4	0.0800	0.0800	0.0000	0.7200	0.7200	0.0000
0.5	0.1000	0.1000	0.0000	0.8998	0.9000	4.000e-08
0.6	0.1200	0.1200	0.0000	1.0794	1.0800	3.6000e-07
0.7	0.1400	0.1400	0.0000	1.2582	1.2600	3.240e-06
0.8	0.1599	0.1600	1.0000e-08	1.4356	1.4400	1.936e-05
0.9	0.1799	0.1800	1.0000e-08	1.6104	1.6200	9.216e-05
1	0.1998	0.2000	4.0000e-08	1.7809	1.8000	3.6481e-04

Table 2: Euler predictor-corrector expansion with fuzzy Simpson method, for $\rho = 0.2$ for example1

$\boldsymbol{\omega}$	$\underline{y}$	Exact Solution of $\underline{y}$	Least Square error	$\overline{y}$	Exact Solution of $\overline{y}$	Least Squares error
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.0200	0.0200	0.0000	0.1800	0.1800	0.0000
0.2	0.0400	0.0400	0.0000	0.3600	0.3600	0.0000
0.3	0.0600	0.0600	0.0000	0.5400	0.5400	0.0000
0.4	0.0800	0.0800	0.0000	0.7200	0.7200	0.0000
0.5	0.1000	0.1000	0.0000	0.8999	0.9000	0.0000
0.6	0.1200	0.1200	0.0000	1.0798	1.0800	1.0000e-09
0.7	0.1400	0.1400	0.0000	1.2593	1.2600	4.0000e-08
0.8	0.1600	0.1600	0.0000	1.4393	1.4400	4.9000e-07
0.9	0.1800	0.1800	0.0000	1.6185	1.6200	2.2500e-06
1	0.2000	0.2000	0.0000	1.7976	1.8000	5.7600e-06



(a) **(b)**

Figure 1: Comparison of the exact and obtained numerical solutions using Euler predictor-corrector with fuzzy Trapezoidal method and fuzzy Simpson method in (a, b) respectively for $\rho = 0.2$ for example 1.

EXAMPIE .2 Consider the following second order fuzzy nonlinear Volterra integro differential equation:

$$\underline{y}''(\boldsymbol{\omega},\rho) = 2 - \frac{\boldsymbol{\omega}^7}{18} + \frac{(1-\rho)\boldsymbol{\omega}^5}{36} - \frac{(1-\rho)^2\boldsymbol{\omega}^3}{216} + \int_0^{\omega} \frac{s\omega}{3}\underline{y}^2(s,\rho)ds,$$

$$\underline{y}''(\boldsymbol{\omega},\rho) = 2 - \frac{\boldsymbol{\omega}^7}{18} + \frac{(1-\rho)\boldsymbol{\omega}^5}{36} + \frac{(1-\rho)^2\boldsymbol{\omega}^3}{216} + \int_0^{\omega} \frac{s\omega}{3}\underline{y}^2(s,\rho)ds \qquad (22)$$

with initial conditions $y'(0,\ \rho) = (0\,,0)$, $y(0,\ \rho)\ = (-\frac{(1-\rho)}{6}\ ,\ \frac{(1-\rho)}{6})$, $0\ \leqslant \boldsymbol{\omega}, s\ \leqslant\ 1$ and $0 \leqslant \rho \leqslant 1$. The exact solution of given fuzzy integro differential solution is $y(\boldsymbol{\omega},\rho) = (\boldsymbol{\omega}^2 - \frac{(1-\rho)}{6}, \boldsymbol{\omega}^2 + \frac{(1-\rho)}{6})$. The numerical results and its graphical visualization are displayed by Tables III and IIV and Fig. II, respectively.

Table 3: Taylor series expansion with fuzzy Trapezoidal method, for $\rho = 0.2$ for example 2

ω	$\underline{y}$	Exact Solution of $\underline{y}$	Least Squares error	$\overline{y}$	Exact Solution of $\overline{y}$	Least Squares error
0	-0.1333	-0.1333	0.0000	0.1333	0.1333	0.0000
0.1	-0.1233	-0.1233	0.0000	0.1433	0.1433	0.0000
0.2	-0.0933	-0.0933	0.0000	0.1733	0.1733	0.0000
0.3	-0.0433	-0.0433	0.0000	0.2233	0.2233	0.0000
0.4	0.0267	0.0267	0.0000	0.2933	0.2933	0.0000
0.5	0.1167	0.1167	0.0000	0.3833	0.3833	0.0000
0.6	0.2267	0.2267	0.0000	0.4933	0.4933	0.0000
0.7	0.3567	0.3567	0.0000	0.6233	0.6233	0.0000
0.8	0.5067	0.5067	0.0000	0.7733	0.7733	0.0000
0.9	0.6767	0.6767	0.0000	0.9434	0.9433	0.0001
1	0.8667	0.8667	0.0000	1.1334	1.1333	0.0001

Table 4: Taylor series expansion with fuzzy Simpson method, for $\rho = 0.2$ for example 2

ω	$\underline{y}$	Exact Solution of $\underline{y}$	Least Squares error	$\overline{y}$	Exact Solution of $\overline{y}$	Least Squares error
0	- 0.1333	- 0.1333	0.0000	0.1333	0.1333	0.0000
0.1	- 0.1233	- 0.1233	0.0000	0.1433	0.1433	0.0000
0.2	- 0.0933	- 0.0933	0.0000	0.1733	0.1733	0.0000
0.3	- 0.0433	- 0.0433	0.0000	0.2233	0.2233	0.0000
0.4	0.0267	0.0267	0.0000	0.2933	0.2933	0.0000
0.5	0.1167	0.1167	0.0000	0.3833	0.3833	0.0000
0.6	0.2267	0.2267	0.0000	0.4933	0.4933	0.0000
0.7	0.3567	0.3567	0.0000	0.6233	0.6233	0.0000
0.8	0.5067	0.5067	0.0000	0.7733	0.7733	0.0000
0.9	0.6766	0.6767	0.0001	0.9433	0.9433	0.0000
1	0.8666	0.8667	0.0001	1.1332	1.1333	0.0001

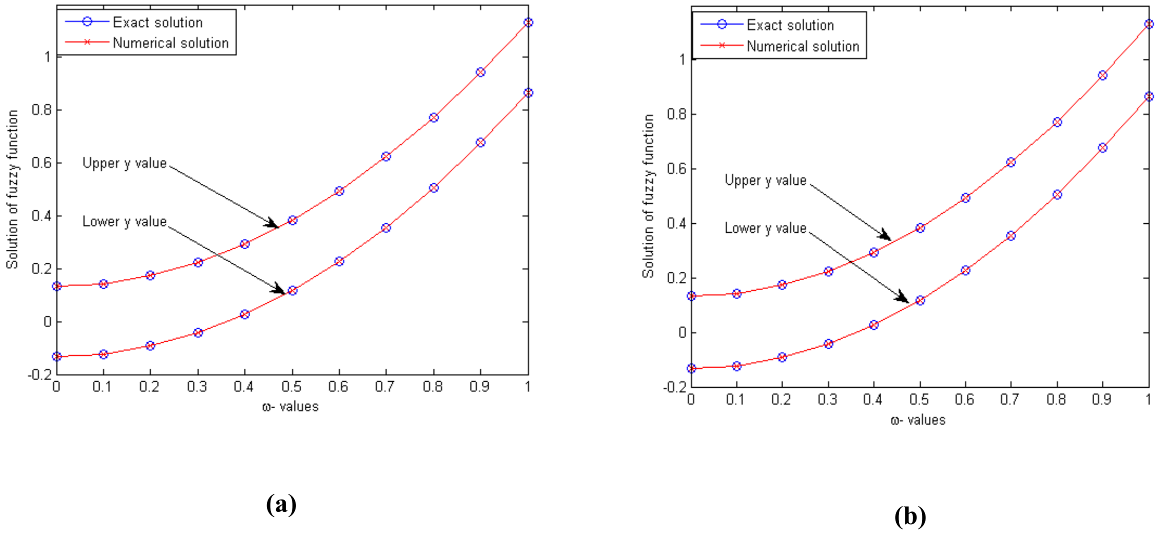


Figure 2: Comparison of the exact and obtained numerical solutions using Euler predictor-corrector with fuzzy Trapezoidal method and fuzzy Simpson method in (a, b), respectively, for $\rho = 0.2$ for example 2.

6. CONCLUSION

In this research, we suggested a combination of two numerical approaches to solve first- and second-order fuzzy nonlinear volterra integro-differential equations. This proposal uses fuzzy Taylor expansion with fuzzy Newton-Cotes integration and fuzzy Euler predictor-corrector with fuzzy Newton-Cotes integration to handle first- and second-order equations, respectively. The suggested method was tested using two distinct scenarios, and the tables and figures provide compelling findings. Finally, we conclude that our study offers a more straightforward and efficient method for solving fuzzy nonlinear integral differential equations.

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