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$ic-$ Separation Axioms in Topological Spaces

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Abstract

A new class of separation axioms known as ic-separation axioms is introduced in this study. They rely on new extended open sets known as ic-open sets, and we describe the relationship between them and give several examples. Additionally, we define ic-generalized closed set concepts in a topological space in order to frame the another class of separation axioms called ic-generalized separation axioms. Among other things, the basic concern properties and relative preservation properties of these spaces are projected under ic-generalized irresolute mappings.

Keywords: $ic-$ open sets, $ic-$ continuous map, ic-irresolute map, Separation Axioms.

بديهيات الفصل من النمط - ic في الفضاء التبولوجي

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الخلاصة:

تم تقديم فئة جديدة من بديهيات الفصل تعرف بديهيات الفصل من النمط - ic في هذة الدراسة. انها تعتمد على مجموعات مفتوحة موسعة جديدة تعرف باسم مجموعات مفتوحة من النمط – ic، ونحن نصف العلاقة بينها ونقدم العديد من الامثلة.

بالاضافة الى ذلك، نحدد مفاهيم المجموعة المغلقة المعممة من النمط – ic في القضاء التبولوجي من اجل خلق فئة اخرى من بديهيات الفصل تسمى بديهيات الفصل من النمط - ic المعمم. من بين اشياء اخرى، يتم عرض الخصائص الاساسية للمجموعة وخصائص النسبية لهذة الفضاءات في اطار التطبيقات المذبذبة.

الكلمات المفتاحية: المجموعات المفتوحة من النمط - ic ، التطبيق المستمر من النمط – ic ، التطبيق المذبذب من النمط - ic، بديهيات الفصل.

Introduction

By utilizing the idea of pre-open sets, Fatima, M. Mohammad [1] established pre-Techonov and pre-Hausdorff separation axioms in Intuitionistic Fuzzy Special Topological Spaces in 2006. Another sort of separation axioms dependent on i-open sets was presented in 2016 by Sabih W. Askandar [2]. The purpose of this study is to present a novel separation axiom that depends on ic-open sets., which we call $ic-$ separation axioms such as (T_{0ic}, T_{1ic}, T_{2ic}, $ic-regular\ and\ ic-normal\ space$). This collection of separation axioms can be used in connection with others to compare and discover characteristics and features that are comparable. Also, the concept of an *ic*-generalized closed set has been coined and then *ic*-generalized separation axioms have been framed with respect to *ic*-generalized open sets. We denoted the topological spaces (X,τ) and (Y,σ) simbly by X and Y respectively, open sets (resp. closed sets) by (*os*), (*cs*) and the phrase topological space by *TS*. (X, τ^{ic}) and (X, τ^{icg}) are always topological spaces throughout this work, where τ^{ic} and τ^{icg} represent the family of all *ic*-open and *icg*-open sets of X.

1. Preliminaries

Throughout this paper $cl(E)$ and $Int(E)$ respectively closure and the interior of the set E, where E is a subset of a topological space (X,τ) on which no separation axioms are assumed unless explicitly stated.

Definition 1.1: If E be a subset of a space X, then E is named

(1) $ic-$open set [3] is denoted by (ic-os) if there exists closed set $F \neq \phi,\ X \in \tau^c$ such that: $F \cap E \subseteq Int(E)$, where $Int(E)$ is the interior of E and τ^c is the family of all (cs), the complement of ic-open is said ic-closed and designated by (ic-cs).
(2) g-closed set [4] is designated by (g-cs) if $cl(E) \subset U$ whenever $E \subset U$ and U is (os).
(3) $ico(X)$, $icc(X)$ and $gc(X)$ are family of ic-open, ic-closed and g-closed sets respectively.

Definition 1.2: (1) The $ic-$closure of a subset E of X [3] is the intersection of all ($ic-$cs) that contains E and is denoted by $cl_{ic}(E)$.

(2) The $ic-$interior of a subset E of X is the union of all (ic-os) subsets of X [3] that contained in E and is denoted by $Int_{ic}(E)$.

Definition 1.3: A mapping $f : X \to Y$ is called:

1. "Continuous" denoted by *(contm)*[3], if $f^{-1}(F)$ is (cs) in X. $\forall F \in (cs)$ in Y.
2. "ic-continuous" denoted by (ic-$contm$) [3], if $f^{-1}(F)$ is (ic-cs) in X.$\forall F \in (cs)$ in Y.

Theorem 1.4. 1. *Each (cs) in TS is (g-cs) [4].* **2.** *Each (os) in TS is (ic-os) [3].*

Definition 1.5: "A TS(X, τ) is said to be:
(1) $T_{0ic}-$space iff [3] to each set of unique points a, b of X, there exists an(ic-os) containing one but not the other.

(2) $T_{1ic}-$space iff [3] to each set of unique points a, b of X, there exists a pair of (ic-os), one containing a but not b, and the other with b. but not a".

(3) "$T_{2ic}-$space iff [3] to each pair of distinct points a, b of X, there exists a pair of disjoint(ic-os), one containing a and the other with b".

(4) $T_{1/2}$-space [5] if each (g-cs) is (cs).

Theorem 1.6. *[3] Each $T_{2ic}-$space is T_{1ic} and also is T_{0ic} .*

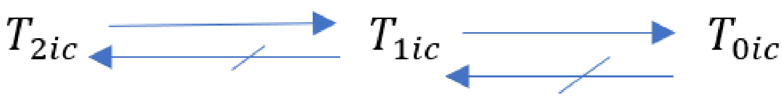

Figure 1

2. $ic-$Separation Axioms in Topological Spaces.

In this section, we discuss and study new class of separation axioms by utilizing ic-open set.

Definition 2.1: A TS (X, τ) is said to be:

(1) ic-regular space: If for every $(cs)F$ and each point p of X which is not in F, there exist disjoint *(ic-os)* H and G s.t. $p \in H$, and $F \subseteq G$.
(2) ic-normal space: If for every pair of disjoint (cs) F_1 and F_2 in X, there exists disjoint *(ic-os)* H and G such that $F_1 \subset H$ and $F_2 \subset G$.

Example 2.2:

Let $= \{5,7\}$, $\tau = \{\emptyset, X, \{5\}, \{7\}\}$, then

$\tau^{ic} = \{\emptyset, X, \{5\}, \{7\}\} = c(\tau^{ic})$

(X, τ) and (X, τ^{ic}) are topological spaces.

1. $5,7 \in X$ $(5 \neq 7)$ $\exists$ $\{5\}, \{7\} \in \tau^{ic}$, such that $5 \in \{5\}$, $7 \in \{7\}$. Therefore; (X, τ) is a $T_{1ic} - space$.
2. $5, 7 \in X$ $(5 \neq 7)$ $\exists$ $\{5\}, \{7\} \in \tau^{ic}$ s.t. $5 \in \{5\}, 7 \in \{7\}, \{5\} \cap \{7\} = \emptyset$. Therefore; (X, τ) is aT_{2ic}.
3. $\{7\}$ is an ic-closed set and $5 \notin \{7\}$ there are two ic-open sets $\{5\}$,$\{7\}$ s.t. $5 \in \{5\}, \{7\} \subseteq \{7\}$. Therefore; (X, τ) is a ic-regular space.
4. $\{5\}$, $\{7\}$ are ic-closed sets there are two ic-open sets $\{5\}$, $\{7\}$ s.t. $\{5\} \subseteq \{5\}, \{7\} \subseteq \{7\}$, $\{5\} \cap \{7\} = \emptyset$. Therefore; (X, τ) is a ic-normal space.

Theorem 2.3: *A space X is a*T_{0ic} *iff* $cl_{ic}\{x\} \neq cl_{ic}\{y\}$ *for each individual pair of points x, y of X.*

Proof: The proof is obtained by the same way of proving (Theorem 4.3[2]).■

Theorem 2.4: *A space X is* T_{1ic} *iff the singleton sets are ic-closed sets.*

Proof: Let X be aT_{1ic} −space and let $m \in X$, to prove that $\{m\}$ is $(ic{-}cs)$, we will show $X \backslash \{m\}$ is $(ic{-}os)$ in X. Let $n \in X \backslash \{m\}$, implies $m \neq n$ and since X is a T_{1ic} −space then, their exist two ic−open sets W, Z s.t. $m \notin W$, $n \in Z \subset X \backslash \{m\}$. Since $n \in Z \subset X \backslash \{m\}$ then $X \backslash \{m\}$ is *(ic*−os). Hence $\{m\}$ is *(ic*−cs).

Conversely, let $m \neq n \in X$, and then $\{m\}, \{n\}$ are $(ic{-}cs)$. That is $X \backslash \{m\}$ is $(ic{-}os)$, clearly, $m \notin X \backslash \{m\}$ and $n \in X \backslash \{m\}$. Similarly, $X \backslash \{n\}$ is $(ic{-}os)$, clearly, $n \notin X \backslash \{n\}$ and $m \in X \backslash \{n\}$. Hence X is a T_{1ic} −space. ■

Theorem 2.5: *A space (X,τ) is a T_{2ic} −space iff (X,τ^{ic}) is a "Hausdorff –space".*

Proof: Suppose that s, $r\in X$ with $s\neq r$. Since X is a T_{2ic} −space, there exist disjoint (*ic-os*) W and Z in X s.t. $s\in W$ and $r\in Z$, $W\cap Z=\emptyset$. Here W, $Z\in\tau^{ic}$, so, it is evident that (X,τ^{ic}) is no longer a Hausdorff space or a T_{2ic} -space.

Conversely, whenever (X,τ^{ic}) is a T_{2ic} −space , there exists a pair of members of τ^{ic}, say, K & F regarding two separate points s & r of X s.t. $s\in K$ & $r\in F$ & $K\cap F=\emptyset$. But $ico(X,\tau)=\tau^{ic}$. Combing all these facts (X,τ) is a T_{2ic} *−space* ■

Theorem 2.6: *Every open subspace of a T_{2ic} −space is T_{2ic} .*

Proof: Assume that (X,τ) *be*T_{2ic} − space *and* W be an open subspace of it. Let s and r represent any two separate points on W. Since X is a T_{2ic} −space and $W\subset X$, there is two separate "(*ic-os*)" Q and L in X s.t. $s\in Q$ & $r\in L$. Let $I=W\cap Q$ & $J=W\cap L$. Then I & J are (*ic-os*) in W containing s and r. Also, $I\cap J=\emptyset$. Hence (W,T_u) is T_{2ic}. ■

Theorem 2.7: *Each regular space is ic-regular.*

Proof: Assume that (X,τ) be a regular space. Let $k\in X$ and F be any closed set on X, s.t. $k\notin F$, and let U, V be any (*os*) in X, s.t. $U\cap V=\emptyset$. Form Theorem 2.4 (2) each (*cs*) is (*ic-cs*). Then F is (*ic-cs*) & $k\notin F$. Because every "(os)" is a "(ic-os),", then U and V is (*ic-os*). Hence (X,τ) is *ic*-regular because $F\subseteq V$ *and* $k\in U$. ■

Illustration 2.8.

Let $X=\{1,2,3\}$, $\tau=\{\emptyset,X,\{1\},\{1,3\}$. Then X is ic − regular space but not a regular space.

Theorem 2.9. *Each normal space is a ic*-normal− *space* but not conversely.

Proof: Suppose that (X,τ) be a normal space and F_1, F_2 be disjoint (*cs*) in X and U, V are disjoint (*os*) in X , s.t. $F_1\subset U$, $F_2\subset V$ and $U\cap V=\emptyset$. Form Theorem 2.4(2) each (*cs*) is (*ic-cs*) and every (*os*) is (*ic-os*) in X. Then F_1, F_2 are (*ic-cs*) and U, V are (*ic-os*). Hence (X,τ) is *ic*-normal . ■

If $X=\{1,2,3\}$, $\tau=\{\emptyset,X,\{2\},\{1,2\},\{2,3\}\}$.

Then X is a $ic-normal$ space but not a normal space.

3. Invariant property of T_{kic} spaces (k=0, 1, 2).

We, now, introduce the invariant property of the T_{kic} spaces in the following manner:

Definition 3.1: A mapping $f: X \to Y$ is called *ic*-irresolute is designated by (*ic-irrem*), if f^{-1} *(F)*is (*ic-cs*) set in X. $\forall$ $F \in (ic - cs)$ in Y.

Example 3.2: Let $X = Y = \{1, 2, 3\}$,

$\tau = \{\emptyset, X, \{2\}, \{1, 2\}, \{2, 3\}\}$, $\sigma = \{\emptyset, Y, \{3\}\}$

Let $f: X \to$ be the identity map. Then f is *ic*-irresolute mapping.

Theorem 3.3: If $f: X \to Y$ *be injective, (ic-irrem) and Y is* $T_{0ic} - space$*, then X is a* $T_{0ic} - space$.

Proof: Assumes that *n*, *m* $\in X$, *n*$\neq$ *m*. Because f is injective and *Y* is a T_{0ic} − space there exists (*ic- os*) *U* in *Y* s.t. *f(n)* ∈*U* and *f(m)* ∉*U* or there exists (*ic- os*) *G* in *Y* s.t. *f(m)* ∈*G* and *f(n)* ∉*G* with *f(n)* ≠ *f(m)*. By (*ic-irrem*) of *f*, $f^{-1}(U)$ is (*ic-os*) in *X* s.t. $n \in f^{-1}(U)$ and $m \notin f^{-1}(U)$ *or* $f^{-1}(G)$ is (*ic-os*) in *X* s.t. $m \in f^{-1}(G)$ and $n \notin f^{-1}(G)$. This shows that *X* is T_{0ic} − space. ■

Theorem 3.4: *If* $f: X \to Y$ *be injective, (ic-irrem) and Y is a* $T_{1ic} - space$*, then X is a* $T_{1ic} - space$.

Proof: The argument is valid in the manner suggested by Theorem 4.3 with suitable changes.

Theorem 3.5: *If* $f: X \to Y$ *is an injective and (ic-irrem) map and Y is* $T_{2ic} - space$*, then X is* $T_{2ic} - space$.

Proof: Similarly to proof of Theorem 4.3 for the establishment of the statement of the theorem under proper changes according to the context.

Theorem 3.6:. *If* (X, τ) *is assumed to be TS, then the subsequent arguments are related.*

1- *X is a* $T_{2ic} - space$.
2- *Let* $k \in X$ *for every k≠ p, there exists(ic-os) U containing k s.t. p*$\notin cl_{ic}(U)$.
3- *For each* $k \in X$ $\cap\{cl_{ic}(U): U \in ico(X)$ & $k \in U\}=\{k\}$

Proof:(1)$\rightarrow$(2): Assumes (X, τ) is $T_{2ic} -$ space, there exist disjoint (*ic-os*) U and G containing k and p respectively. So, $U \subset X \backslash G$. Therefore; $cl_{ic}(U) \subset X \backslash G$. So $p \notin cl_{ic}(U)$.

(2) $\rightarrow$(3): If possible for some $k \neq p$, we have $p \in cl_{ic}(U)$ for every (*ic-os*) *U containing k*, which is contradiction (2).

(3) $\rightarrow$(1): Suppose $k, p \in X$ **&** $k \neq p$. Then there exists *(ic-os) U containing k s.t.* $p \notin cl_{ic}(U)$. Let $G = X \backslash cl_{ic}(U)$, then $p \in G$ and $k \in U$ and also $U \cap G = \emptyset$.■

Definition3.7: A mapping $f: X \rightarrow Y$ is said to be point *ic*-closure 1-1 iff $n, m \in X$ such that $CL_{ic}\{n\} \neq CL_{ic}\{m\}$ then $CL_{ic}\{f(n)\} \neq CL_{ic}\{f(m)\}$

Example 3.8: Let $X = Y = \{2,3\}$, $\tau = \{\emptyset, X, \{2\}, \{3\}\}, \sigma = \{\emptyset, Y, \{2\}\}$

Let $f: X \rightarrow Y$ be the identity map. Then f is point *ic*-closure 1-1, because $2, 3 \in X$ such that $CL_{ic}\{2\} \neq CL_{ic}\{3\}$ then $CL_{ic}\{f(2)\} \neq CL_{ic}\{f(3)\}$

Theorem3.9: *If $f: X \rightarrow Y$ is point ic-closure 1-1 and X is a $T_{0ic} - space$, then f is one to one mapping.*

Proof: Suppose $n, m \in \chi$ with $n \neq m$. Since χ is $T_{0ic} -$ space, then $CL_{ic}\{n\} \neq CL_{ic}\{m\}$ by Theorem 3.3. But f is point *ic*-closure 1-1 implies that $CL_{ic}\{f(n)\} \neq CL_{ic}\{f(m)\}$. Hence $f(n) \neq f(m)$ Thus, f is one to one mapping. ■

Theorem 3.10: ***A*** *point ic-closure 1-1 mapping $f: X \rightarrow Y$ from T_{0ic}-space X into $T_{0ic} - spac$ Y exists iff f is one to one mapping.*

Proof: The necessity follows from the fact mentioned in Theorem 4.3 For sufficiency, let $f: X \rightarrow Y$ from T_{0ic}-space X into a $T_{0ic} -$ spac Y be an 1-1 mapping. Now for every pair of distinct points n & $m \in X$, $CL_{ic}\{n\} \neq CL_{ic}\{m\}$ *as* X is a $T_{0ic} -$ space. Since, f is 1-1 mapping $f(CL_{ic}\{n\}) \neq f(CL_{ic}\{m\})$.i.e., $CL_{ic}\{f(n)\} \neq CL_{ic}\{f(m)\}$. Consequently, f is point *ic*-closure 1-1mapping. ■

4. *ic*-Generalized Separation Axioms.

Separation axioms using *ic* generalized-open sets and being more than *ic*-separation axioms are, here, framed due to the motivation of the existence & wide application of *ic*-generalized open sets.

Definition 4.1: (1) If E be a subset of a space X then E is supposedly an ic-generalized closed set is denoted by (*icg-cs*) if $cl_{ic}(E) \subset U$ whenever $E \subset U$ and U is (*ic*-os), the set of all family (*icg*-cs) denoted by *icg c(X)* .

(2)The $icg -$ closure of a subset E of X is the intersection of all $(icg -$ cs$)$ that contains E and is denoted by $cl_{icg}(E)$.

(3) The $icg -$ interior of a subset E of X is the union of all (*icg-os*) subsets of X that contained in E and is denoted by $Int_{icg}(E)$.

Example 4.2: Let $X = \{1, 2, 3\}$ and let $\tau = \{\emptyset, X, \{2\}, \{1, 2\}\}$. Then

$c(X) = \{\emptyset, X, \{1, 3\}, \{3\}\}$.

$ic\ o(X) = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}\}$.

$ic\ c(X) = \{\emptyset, X, \{2, 3\}, \{1, 3\}, \{3\}\} = icg\ c(X)$.

Theorem 4.3: *Each (cs) in space X is (icg-cs).*

Proof: Let E be (*cs*) in X s.t. $E \subset U$, where U is (*ic-os*). Since E is closed, then $(E) = E$, since $cl_{ic}(E) \subset cl(E) = E$, and $E \subset U$, therefore; $cl_{ic}(E) \subset U$. Hence E is *(icg-cs) in X.* ■

From example (5.2).

Let $A = \{2, 3\}$. Here A is (*icg-cs*) but not (*cs*).

Theorem 4.4: *Each (ic-cs) in space X is (icg-cs) but not conversely.*

Proof: Let E be (*ic-cs*) in X s.t. $E \subset U$, where U is (*ic-os*). Since E is (*ic-cs*), then $cl_{ic}(E) = E$, and $E \subset U$, therefore; $cl_{ic}(E) \subset U$. Hence E is *(icg-cs) in X.* ■

Illustration4.5.

Let $X = \{1, 2, 3\}$and let $\tau = \{\emptyset, X, \{2\}, \{1, 2\}, \{2, 3\}\}$, then

$c(X) = \{\emptyset, X, \{1, 3\}, \{3\}, \{1\}\}$.

$ic\ o(X) = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}\}$.

$ic\ c(X) = \{\emptyset, X, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\}, \{1\}\}$.

$icg\ c(X) = \{\emptyset, X, \{2, 3\}, \{1, 3\}, \{1, 2\}, \{3\}, \{1\}, \{2\}\}$.

Let *A*= $\{2\}$. Then *A* is (*icg-cs*) but not (*ic-cs*).

Theorem 4.6. *Each (g-cs) in space X is (icg-cs) but not conversely.*

Proof: Let *E* be (*g-cs*) in *X* s.t. $E \subset U$ and $cl(E) \subset U$, where *U* is (*os*). Since $cl_{ic}(E) \subset cl(E) = E$, we get $cl_{ic}(E) \subset U$, because every "(*os*)" is "(*icg-os*)", we conclude that *U* is (*ic-os*). Henceforth *E* is *(icg-cs) in X*. ■

Note 4.7.

Let consider Illustration 5.5. We have $icg\ c(X) = \{\emptyset, X, \{2,3\}, \{1,3\}, \{1,2\}, \{3\}, \{1\}, \{2\}\}$.

$gc(X) = \{\emptyset, X, \{1,3\}, \{3\}, \{1\}\}$.

Suppose *A*= {2, 3}. Then *A* is (*icg-cs*) but not (*gc-cs*).

Definition 4.8. A space (X, τ) is called

(1) *ic*- generalized-T_0 (briefly written as T_{0icg}) iff to each pair of "distinct points" a, b of X, there exists an *icg*-open set containing one but not the other.

(2) *ic*- generalized-T_1 (briefly written as T_{1icg}) iff to each pair of "distinct points" a, b of X, there exists a pair of *icg*-open sets, one containing a but not b, and their other containing b but not a.

(3) *ic*- generalized-T_2 (briefly written as T_{2icg}) iff to each pair of "distinct points" a, b of X, there exists a pair of disjoint *icg*-open sets, one containing a and the other containing b".

(4) *icg*-regular space: if for every (cs) F and each point *p* of X which is not in F, there exists disjoint *(icg*-os) *H* and *G* s.t. $p \in H$, and $F \subseteq G$.

(5) *icg*-normal space: if for every pair of disjoint (cs) F_1 and F_2 in X, there is separated *(icg*-os) *H* and *G* s.t. $F_1 \subset H$ and $F_2 \subset G$.

Theorem 4.9. *Each* T_0 *– space is* T_{0icg} *–space.*

Proof: Let *X* be a T_0 –space. Let *a*, *b* be two distinct points in *X*. Since *X* is T_0-space, there exists an (*os*) *U* in *X* s.t. $a \in U$, $b \notin U$. Because every "(*os*)" is "(*icg-os*)", *U* is an (*icg-os*) in *X* containing *a* not *b*. Hence *X* is T_{0icg} *–space.* ■

Let *X* ={*n*, m, k}and let τ ={Ø, *X*, {n, m}}. Then by Definition 5.1, we have

$icg\ o(X) = \{\emptyset, X, \{n\}, \{m\}, \{n, m\}\}$.

Therefore (X, τ) is not a T_0 space, but $icgo(X)$ is a T_{0icg} -space.

Theorem 4.10 *Each T_1– space is a T_{0icg}–space but not the opposite.*

Proof: Since each T_1–space is T_0 and every T_0–space is a T_{0icg}. Therefore; every T_1–space is a T_{0icg}. ■

We illustrate the above theorem by supposing $X = \{2, 3, 4\}$and let $\tau = \{\emptyset, X, \{2\}, \{2, 3\}\}$. Then (X, τ) is not a T_1- space, but $icgo(X)$ is T_{0icg}

Theorem 4.11. *Each T_1–space is a T_{1icg}–space but not the opposite.*

Proof: Assumes X be a T_1 –space and a, b be two separate X points. Because X is a T_1-space, there exists two (*os*) U, V in X s.t. $a \in U, b \notin U, b \in V, a \notin V$. Since each (*os*) is (*icg-os*), U , V are (*icg-os*) in X. Hence X is a T_{1icg}*–space.* ■

Example 4.12. Let $X = \{a, b, c\}$and let $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is not a T_1-space, but $icgo(X)$ is a T_{1icg} - space.

Theorem 4.13. *Each T_2 – space is T_{2icg} but not the opposite.*

Proof. Suppose that X be a T_2 - space and a, b be two separate points in X. Since X is a T_2-space. Then there are two disjoint (*os*) U and V containing a and b respectively. Since each *(os)* is an *(icg-os*). Then U and V are separated (*icg-os*) containing a and b respectively. Hence X is a T_{2icg} - space. ■

Example 5.14. Let $X = \{1, 2, 3\}$ and let $\tau = \{\emptyset, X, \{1\}, \{2, 3\}\}$. Then (X, τ) is not a T_2 -space, but $ico(X)$ is T_{2icg} .

Theorem 4.15. *Each T_2–space is T_{1icg} .*

Proof: Since eachT_2–space is T_1 and eachT_1–space is a T_{1icg} . Therefore; every T_2–space is a T_{1icg}–space. ■

Theorem 4.16. *A space (X, τ) is T_{2icg} –space iff (X, τ^{icg}) is Hausdorff -space.*

Proof: Assumes $s, r \in X$ with $s \neq r$. Since X is T_{2icg} –space, there exists disjoint (*icg-os*) W & Z in X s.t. $s \in W$ & $r \in Z$, $W \cap Z = \emptyset$. Here $W, Z \in \tau^{icg}$, so, obviously (X, τ^{icg}) ceases to be a T_{2icg} –space i.e. a Hausdorff space. ■

Conversely, whenever (X, τ^{icg}) is a T_{2icg} − space, there exists a pair of members of τ^{icg}, say, M & N for a pair of distinct points s & r of X such that $s \in M$ & $r \in N$ & $M \cap N = \emptyset$. But $icgo(X, \tau) = \tau^{icg}$. Combing all these facts (X, τ) is T_{2icg} −*space* ■

Theorem 4.17. *Each open subspace of a* T_{2icg} −*space is* T_{2icg} .

Proof: Suppose W be an open subspace of a T_{2icg} −space (X, τ). Let s and r be any two distinct points of W. Since X is T_{2icg} −space and $W \subset X$, there exists two disjoint (*icg-os*) Q and J in X such that $s \in G$ & $r \in H$. Let $Q = W \cap G$ & $J = W \cap H$. Then Q & J are (*icg-os*) in W containing s and r. Also, $Q \cap J = \emptyset$. Hence (W, T_u) is T_{2icg}. ■

Theorem 4.18. *Each "regular spac" is icg-regular* .

Proof: Assume that (X, τ) be a regular space. Let $k \in X$ and F be any (cs) set on X, s.t. $k \notin F$, and let U, V be any (*os*) in X , s.t. $U \cap V = \emptyset$. Form Theorem 5.3 each (*cs*) is (*icg-cs)*. Then F is (*icg-cs*) &$k \notin F$. Since each "(*os*)" is "(*icg-os*)", then U and V are (*icg-os)*. Hence (X, τ) is *icg*-regular. ■

Theorem 4.19. *Each normal space is an icg*-normal .

Proof: Suppose that (X, τ) is a normal space and F_1 and F_2 is disjoint (*cs*) in X . Let U *and* V be disjoint (*os*) in X , s.t. $F_1 \subset U$, $F_2 \subset V$ and $U \cap V = \emptyset$. Form Theorem 5.3 each (*cs*) is (*icg-cs*) and every (*os*) is (*icg-os*) in X. Then F_1, F_2 are (*icg-cs*), also U *and* V are (*icg-os*). Hence (X, τ) is *icg*-normal . ■

Clearly, each T_{kic}–space is T_{kicg}–spaces (k=0, 1, 2) since each (*ic-os*) is (*icg-os*).

Remark 4.20. *By the above results we have the following diagram.*

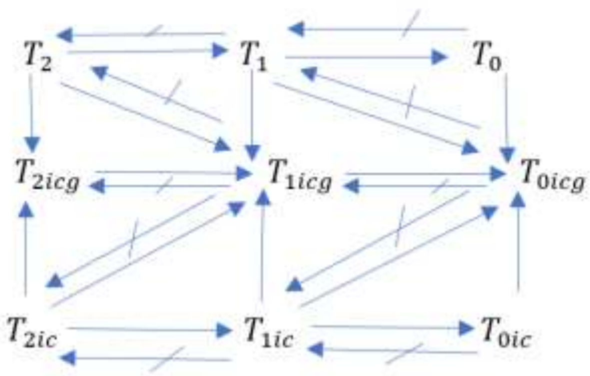

Figure 2

The following theorems are related to the characterization & invariance nature for T_{kicg}-spaces (k=0, 1, 2).

Theorem 4.21. *A space X is T_{0icg} iff $cl_{icg}\{x\} \neq cl_{icg}\{y\}$ for every pair of distinct points x, y of X.*

Theorem 4.22. *A space X is T_{1icg} iff the singletons are icg-closed sets.*

Theorem 4.23. *Let (X,τ) be TS then the following statements are equivalent.*

1- *X is a $T_{2icg}-space$.*

2- *Let $k \in X$.For each $k \neq p$, there exists(icg-os) U containing k s.t. $p \notin cl_{icg}(U)$.*

3- *For each $k \in X$ $\cap\{cl_{icg}(U)$: $U \in \tau^{icg}$ & $k \in U\}=\{k\}$*

Theorem 4.24. *If $f: X \rightarrow Y$ be an injective (icg- irrem) and Y isT_{kicg} then X is T_{kicg} (k=0, 1, 2).*

Definition 4.25. A mapping $f: X \rightarrow Y$ is said to be point *icg*-closure 1-1 iff $n, m \in \chi$ such that $CL_{icg}\{n\} \neq CL_{icg}\{m\}$ then $CL_{icg}\{f(n)\} \neq CL_{icg}\{f(m)\}$

Theorem 4.26. *If $f: X \rightarrow Y$ is point icg-closure1-1and X is a $T_{0icg}-space$,then f is 1-1.*

Theorem 4.27. ***A** point icg-closure 1-1mapping $f: X \rightarrow Y$ from a T_{0icg}-space X into a $T_{0icg}-space$ Y exists iff f is one to one.*

Furthermore, we mention the concept of $T_{\frac{1}{2}ic}$ space in the same tune of $T_{\frac{1}{2}}$ space in topology.

Definition 4.28. A space (X,τ) is named

1. $T_{\frac{1}{2}ic}$ space if each (*icg-cs*) is (*ic-cs*)
2. T_{ic}- space if each *(ic-cs*) in it is (*cs*).
3. T_{icg}- space if each (*icg-cs)* in it is (*cs*).

Example 4.29. If X ={3, 4}and $\tau = \{\emptyset, X, \{3\}\}$, $c(\tau) = \{\emptyset, X, \{4\}\}$ then $c(\tau) = ic\ c(X) = icg(X) = \{\emptyset, X, \{4\}\}$

Hence X is a T_{ic}-space and a T_{icg}-space. Also X is $T_{\frac{1}{2}ic}$.

Theorem 4.30. *If (X, τ) is a T_{icg}-space then, for each $k \in X$, {k} is (icg-cs) or open.*

Proof: Let *TS* (X, τ) be a T_{icg}-space. Let $k \in X$, such that$\{k\}$ is not (*icg- cs*) in X. By Theorem 5.3 *{k}* is not (*cs*) in X. So $X\backslash\{k\}$ is not (*os*) in X and X is the only (*os*) containing $X\backslash\{k\}$. So $X\backslash\{k\}$ is *(icg-cs*) in X, by hypothesis, $X\backslash\{k\}$ is (*cs*) in X, it means *{k}* is (*os*) in X. ■

Theorem 4.31. *Each T_{icg}-space is a T_{ic}-space but not the opposite.*

Proof: Suppose (X, τ) is a T_{icg}-space and k is *(ic-cs*) in χ. Since each *(ic-cs*) is (*icg-cs*), therefore; k is (*icg-cs*) in X,by hypothesis, k is (*cs*) in X. This shows that X is a T_{ic}*-space* ■

Example 4.32. Let X ={a, b, c} and $\tau = \{\emptyset, X, \{c\}\}$ then

$c(\tau) = \{\emptyset, X, \{a, b\}\}$.

$ic\ o(X) = \{\emptyset, X, \{c\}\}$.

$ic\ c(X) = \{\emptyset, X, \{a, b\}\}$.

$icg\ c(X) = \{\emptyset, X, \{a, b\}, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$.

Hence (X, τ) is T_{ic}- space. but (X, τ) is not T_{icg}, because {b}is (*icg-cs*) in (X, τ) but {b} is not (*cs*) in (X, τ).

Theorem 4.33. *Each T_{icg}-space is a $T_{1/2}$-space but not the opposite.*

Proof: Suppose (X,τ) be T_{icg}-space and let n be (*g-cs*) in X. Since each (*g-cs*) is (*icg-cs*), therefore; n is (*icg-cs*) in X, by hypothesis, n is (*cs*) in X. This shows that X is $T_{1/2}$. ■

Example 4.34. Let $X = \{1,2,3\}$ and $\tau = \{\emptyset, X, \{2\}, \{1,2\}, \{2,3\}\}$, then

$c(X) = \{\emptyset, X, \{1,3\}, \{3\}, \{1\}\}$.

$ic\ o(X) = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$.

$ic\ c(X) = \{\emptyset, X, \{2,3\}, \{1,3\}, \{1,2\}, \{3\}, \{1\}\}$.

$icg\ c(X) = \{\emptyset, X, \{2,3\}, \{1,3\}, \{1,2\}, \{3\}, \{1\}, \{2\}\}$.

$gc(X) = \{\emptyset, X, \{1,3\}, \{3\}, \{1\}\}$. It is easy to see that (X, τ) is a $T_{1/2}$-*space but not* a T_{icg}-*space*

Remark 4.35. There is no relationship between a T_{icg}-*space* (T_{ic}-*space*,$T_{1/2}$-*space*) and a $T_{\frac{1}{2}ic}$-*space* in a topological space (X, τ) as shown in the following example.

Example 4.36. If $= \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$, then

$c(X) = \{\emptyset, X, \{c\}\}$. $ic\ o(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$.

$ic\ c(X) = \{\emptyset, X, \{b, c\}, \{a, c\}, \{c\}\} = icg\ c(X)$.

It is easy to see that (X, τ) is a $T_{\frac{1}{2}ic}$-*space but it is no at* T_{icg}, T_{ic}-*space and not a* $T_{1/2}$.

Remark 4.37. *From above we have the following diagram:0*

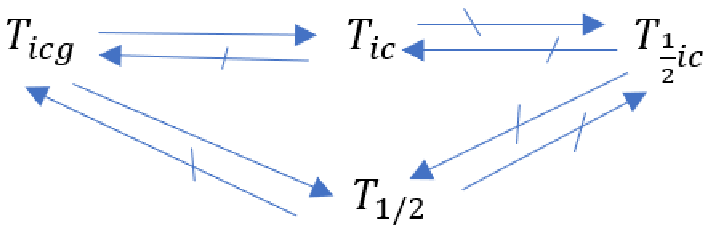

Figure 3

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