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A Posteriori Error Analysis of the FEM Solution for Generic Linear Second-Order ODEs

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Abstract

In this paper, a posteriori error analysis has been examined and investigated for the continuous (conforming) Galerkin finite element method used for solving a general scalar linear second-order ordinary BVPs. Linear elements (piecewise linear polynomials) are used in space discretisation on non-uniform mesh. We derived optimal order a posteriori error bounds in the L_2 norm using the duality approach and standard a posteriori error analysis techniques and tools.

Keywords: A posteriori error analysis, finite element methods, ordinary differential equations.

تحليل الخطأ البعديلحل طريقة العناصر المحدودة للمعادلات التفاضلية الخطية العامة من الدرجة الثانية

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الخلاصة

في هذا البحث، تم فحص ودراسة تحليل الخطأ البعدي لطريقة جاليركين للعناصر المحدودة المستمرة (المطابقة) المستخدمة لحل مسائل القيمة الحدية العادية الخطية من الدرجة الثانية.. تُستخدم العناصر الخطية (متعددة الحدود الخطية) في تقدير المساحة على الشبكات غير المنتظمة. لقد اشتققنا الترتيب الأمثل لحدود الخطأ الخلفي في معيار L_2 باستخدام نهج الازدواجية وتقنيات وأدوات تحليل الخطأ الخلفي القياسي.

الكلمات المفتاحية: تحليل الخطأ البعدي، طرق العناصر المحدودة، المعادلات التفاضلية العادية

Introduction

The Finite Element Method (FEM) is considered as the most powerful and flexible tool that is used for obtaining accurate solutions of simple and complicated ordinary differential equations (ODEs) and partial differential equations (PDEs). The starting point of the method can be traced back to the 1940s and the development of this method passed through many achievements and advancements during the last eight decades. Many of the realistic world problems are modelled by ODEs and PDEs and most of these equations are complicated and have no analytical solutions [17].

The interest in the study of the numerical solutions of ODEs has witnessed an increasing attention of researchers and significant advancements have made. In 1969, Argyris and Scharpf [2] presented and used the variational method for time discretisation for time dependent problems. Hulme [8] is considered the first to use the discontinuous Galerkin (DG) method for solving ODEs in 1972. Delfour et al in 1981 [4] derived a family of DG methods for solving ODEs. Estep [5] in 1995 examined the use of FEM for solving ordinary initial value problems (IVPs) and the author derived optimal order a posteriori and a priori error estimates. Estep and Stuart in 2001 [6] examined and studied the dynamical behaviour of the DG method for ODEs.

In 2014, Janssen and Wihler [10] studied the existence of discrete solutions of the hp - CG and DG time-stepping methods, which are used for solving nonlinear IVPs. In 2014, Zhao and Wei [16] considered and derived a unified DG framework for nonlinear ODEs. In 2016, Holm and Wihler [7] considered and investigated CG and DG time-stepping methods of arbitrary order for solving nonlinear IVPs. Adnan and Ahmed in 2018 in [1] used Galerkin method for solving a system of second order boundary value problems (BVPs) using Bernstein polynomials.
Recently, Schmidt, Beyer, Hinze and Vandoros in 2020 [11] used FEM for solving first order ODEs. In 2022, Huynh [9] used DG method for solving ODEs using the idea of the correction function. Danet [3] in 2022 used the variational methods for solving sixth order ODEs and analysed the existence and uniqueness of their solutions. Sohel et al in 2022 [12] used Galerkin residual correction method for solving fourth order BVPs. Also, the researchers in 2022 in [13] considered using Galerkin residual correction method with modified Legendre polynomials for

approximating the solution of the linear and nonlinear second order BVPs of ODEs. Wang et al in 2023 [15] presented and examined hp-version CG methods for nonlinear second order IVPs of ODEs. In this paper, we derived optimal order a posteriori error estimates in the L_2 norm of the FEM solution of generic linear second-order ordinary BVPs using a conforming Galerkin linear finite element method. This paper is organized as follows: In section 2 we give the necessary notations and relevant preliminaries of the topic. Section 3 is devoted for the a posteriori error analysis for the general scalar linear second-order ordinary BVPs. The conclusions are given in section 4.

Problem Setting and Notation

We consider the problem of finding the solution of the generic scalar linear ordinary BVP: find $u: I \to R$ such that

$$-au'' + bu' + cu = f, \ on\ I, \qquad (1)$$
$$u(\alpha) = 0, u(\beta) = 0,$$

where $I = (\alpha, \beta), a < 0, b, c \geq 0$ and $f \in L_2(I)$. For simplicity of notation let $H = H_0^1(I)$. The weak formulation of the problem in (1) as follows: find $u \in H$ such that $\sigma(u, v) = l(v)$ for all $v \in H$, where

$$\sigma(u, v) = \int_I \quad (au'v' + bu'v + cuv)dx,$$

and

$$l(v) = \int_I \quad fv\, dx = (f, v).$$

Then, there exists a unique weak solution, $u \in H$, where $\sigma: H \times H \to R$ is the bilinear form, $l: H \to R$ is the linear functional associated with this problem and $(.,.)$ *is the* L_2 inner product. The interval $[\alpha, \beta]$ is subdivided into n subintervals (elements) via partition $\Pi: \alpha = x_0 < x_1 < \cdots < x_{n-1} < x_n = \beta$. For simplicity and convenience of exposure, we consider the finite dimensional subspace H_h consists of continuous piecewise linear functions. The finite element approximation of the BVP in (1) is: find $u_h \in H_h$ such that $\sigma(u_h, v_h) = l(v_h)$ for all $v_h \in H_h$. Define$h_i = x_i - x_{i-1}, i = 1, \ldots, n$.

A Posteriori Error Analysis

A posteriori error analysis is a very important and effective tool in designing efficient and effective adaptive methods. It is utilised in finding an estimate or a bound for the error $e = u - u_h$, which depends upon the approximate solution u_h, the source function f and the data of the problem. Our aim is to find an a posteriori estimator function $E = E(u_h, f; V)$ in terms of the functions u_h, f and the space V, such that E satisfies the relation $\| e \|_V = \| u - u_h \|_V \leq E(u_h, f; V)$. The a posteriori error bounds are useful in reducing the computational cost of solving a problem using numerical methods. In this section, we derive the a posteriori error bounds for a generic scalar linear second-order ordinary BVPs using duality technique.

A Posteriori Error Analysis for a Generic Scalar Linear Second- Order ODEs

In this section, we explain how to derive and obtain a posteriori error estimates for general scalar linear second-order ordinary BVPs.

Theorem (L_2 a Posteriori Error Bounds for Generic Scalar Linear Second-Order BVP ODEs) The finite element approximate solution u_h of the problem (1), satisfies the following a posteriori L_2 error estimate

$$\| u - u_h \|_{L_2(I)} \leq C\hat{} h^2 \| R(u_h) \|_{L_2(I)}, \tag{2}$$

where $C\hat{} > 0$ and $R(u_h) = f + au_h'' - bu_h' - cu_h$.

Proof. We consider the following auxiliary dual or adjoint BVP

$$-a\psi'' - b\psi' + c\psi = u - u_h, \; on \; I, \tag{3}$$
$$\psi(\alpha) = \psi(\beta) = 0.$$

We start the proof by writing the dual problem in the weak form by testing it with a test function $u - u_h \in H$ and performing integration by parts to obtain

$$\begin{aligned} \| u - u_h \|_{L_2(I)}^2 &= (u - u_h, u - u_h) = (u - u_h, -a\psi'' - b\psi' + c\psi) \\ &= \sigma(u - u_h, \psi), \end{aligned} \tag{4}$$

noting that $(u - u_h)(\alpha) = 0, (u - u_h)(\beta) = 0$. Using Galerkin orthogonality

$$\sigma(u-u_h,\psi_h)=0, \forall\psi_h \in H_h, \tag{5}$$

and choosing $\psi_h = I_h\psi \in H_h$, where $I_h\psi$ is the continuous piecewise linear interpolant of the function ψ on the partition $\Pi: \alpha = x_0 < x_1 < \cdots < x_{n-1} < x_n = \beta$, which results in

$$\sigma(u-u_h, I_h\psi) = 0.$$

So,

$$\| u-u_h \|^2_{L_2(I)} = \sigma(u-u_h,\psi - I_h\psi) = \sigma(u,\psi - I_h\psi) - \sigma(u_h,\psi - I_h\psi)$$
$$= (f,\psi - I_h\psi) - \sigma(u_h,\psi - I_h\psi). \tag{6}$$

Now, we consider the second term of the right-hand side of (6),

$$\sigma(u_h,\psi - I_h\psi)$$
$$= \sum_{i=1}^{n} a\int_{x_{i-1}}^{x_i} u_h'(\psi - I_h\psi)'dx + \sum_{i=1}^{n} b\int_{x_{i-1}}^{x_i} u_h'(\psi - I_h\psi)dx$$
$$+ \sum_{i=1}^{n} c\int_{x_{i-1}}^{x_i} u_h(\psi - I_h\psi)dx.$$

Performing the integration by parts on the first term of the right-hand side and observing that $(\psi - I_h\psi)(x_i) = 0, i = 0, \ldots, n$, which implies that

$$\sigma(u_h,\psi - I_h\psi) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (-au_h'' + bu_h' + cu_h)(\psi - I_h\psi)dx. \tag{7}$$

Also,

$$(f,\psi - I_h\psi) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(\psi - I_h\psi)dx. \tag{8}$$

Substituting (7) and (8) into (6), we arrive that

$$\| u-u_h \|^2_{L_2(I)} = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} R(u_h)(\psi - I_h\psi)dx, \tag{9}$$

where the function $R(u_h)$ is the finite element residual and it used as an indicator of how the approximate solution u_h fails to satisfy the ODE $-au'' + bu' + cu = f$ on the interval (α,β).

Noting that since u_h is a linear combination of continuous piecewise linear basis functions, then, $u_h'' = 0$ and actually, $R(u_h) = f - bu_h' - cu_h$. Utilising the Cauchy-Schwarz inequality on the right-hand side of (9) leads to

$$\| u - u_h \|_{L_2(I)}^2 \leq \sum_{i=1}^{n} \| R(u_h) \|_{L_2(x_{i-1},x_i)} \| \psi - I_h\psi \|_{L_2(x_{i-1},x_i)} . \quad (10)$$

Using the standard Interpolation Error Bounds [14], we have

$$\| \psi - I_h\psi \|_{L_2(x_{i-1},x_i)} \leq \left(\frac{h_i}{\pi}\right)^2 \| \psi'' \|_{L_2(x_{i-1},x_i)}, i = 1, \dots, n, \quad (11)$$

and by plugging (11) in (10), we find

$$\| u - u_h \|_{L_2(I)}^2 \leq \frac{1}{\pi^2} \sum_{i=1}^{n} h_i^2 \| R(u_h) \|_{L_2(x_{i-1},x_i)} \| \psi'' \|_{L_2(x_{i-1},x_i)}.$$

Finally, we obtain

$$\| u - u_h \|_{L_2(I)}^2 \leq \frac{1}{\pi^2} \left(\sum_{i=1}^{n} h_i^4 \| R(u_h) \|_{L_2(x_{i-1},x_i)}^2 \right)^{\frac{1}{2}} \| \psi'' \|_{L_2(I)} . \quad (12)$$

Now, our aim is to remove ψ'' from the right-hand side of (12). Notice that

$$a\psi'' = u_h - u - b\psi' + c\psi. \quad (13)$$

Hence,

$$\| \psi'' \|_{L_2(I)} \leq \frac{1}{a} \| u - u_h \|_{L_2(I)} + \frac{b}{a} \| \psi' \|_{L_2(I)} + \frac{c}{a} \| \psi \|_{L_2(I)} . \quad (14)$$

Now, we want to bound $\| \psi' \|_{L_2(I)}$ and $\| \psi \|_{L_2(I)}$ in terms of $\| u - u_h \|_{L_2(I)}$ and consequently, $\| \psi'' \|_{L_2(I)}$ by utilising of (14). Consider that

$$(-a\psi'' - b\psi' + c\psi, \psi) = (u - u_h, \psi).$$

Integrating by parts results in

$$(-a\psi'' - b\psi' + c\psi, \psi) = a(\psi', \psi') + b(\psi, \psi') + c(\psi, \psi)$$

$$= a \parallel \psi' \parallel_{L_2(I)}^2 + \frac{1}{2} b \int_I (\psi^2)' dx + c \parallel \psi \parallel_{L_2(I)}^2.$$

observing that $\psi(\alpha) = 0$ and $\psi(\beta) = 0$ and also, integrating by parts in the second term on the right implies

$$(-a\psi'' - b\psi' + c\psi, \psi) = a \parallel \psi' \parallel_{L_2(I)}^2 + c \parallel \psi \parallel_{L_2(I)}^2.$$

Therefore,

$$a \parallel \psi' \parallel_{L_2(I)}^2 + c \parallel \psi \parallel_{L_2(I)}^2 = (u - u_h, \psi),$$

and upon observing that $c \geq 0$, we have

$$\parallel \psi' \parallel_{L_2(I)}^2 \leq \frac{1}{a}(u - u_h, \psi) \leq \frac{1}{a} \parallel u - u_h \parallel_{L_2(I)} \parallel \psi \parallel_{L_2(I)}. \quad (15)$$

Using the Poincaré-Friedrichs inequality [14], we have

$$\parallel \psi \parallel_{L_2(I)}^2 \leq C_{PF} \parallel \psi' \parallel_{L_2(I)}^2, \quad (16)$$

where $C_{PF} = (2/(\beta - \alpha)^2)^{-1}$. Hence, inserting (15) in (16) yields

$$\parallel \psi \parallel_{L_2(I)} \leq \frac{C_{PF}}{a} \parallel u - u_h \parallel_{L_2(I)}. \quad (17)$$

Substituting (17) in (15) results in

$$\parallel \psi' \parallel_{L_2(I)} \leq \frac{C_{PF}^{\frac{1}{2}}}{a} \parallel u - u_h \parallel_{L_2(I)}. \quad (18)$$

Thus, inserting (17) and (18) into (14) to conclude that

$$\parallel \psi'' \parallel_{L_2(I)} \leq C \parallel u - u_h \parallel_{L_2(I)}, \quad (19)$$

where

$$C = \frac{1}{a}\left(1 + bC_{PF}^{1/2} + cC_{PF}\right).$$

Observe that C is computable since it contains only known quantities, the coefficients in the differential equation and the domain of model problem. Inserting (19) into (12), we have

$$\| u - u_h \|_{L_2(I)} \leq C^{\hat{}} \left(\sum_{i=1}^{n} \quad h_i^4 \| R(u_h) \|_{L_2(x_{i-1},x_i)}^2 \right)^{\frac{1}{2}}, \qquad (20)$$

where $C^{\hat{}} = C/\pi^2$. Further, let $h = max_i h_i, i = 1, \dots, n$, and using this in (20), consequently, we obtain our required a posteriori error bound in (2).

Conclusions

We studied the error analysis of the finite element solution of generic scalar linear second-order ordinary BVPs in 1D. Continuous Galerkin finite element method (CGFEM) with piecewise linear polynomials are used for the space discretisation. Optimal order a posteriori error bounds in L_2 norm are obtained using the duality approach and standard a posteriori tools.

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