Academic Science Journal

Numerical Studied for Solving Fuzzy Integro-Differential Equations via Caputo Fractional Derivative

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Received: 11 June 2023 **Accepted: 11 August 2023**

DOI: https://dx.doi.org/10.24237/ASJ.02.04.780B

Abstract

In this paper, we extended some numerical methods to solve fuzzy integro differential equations. The considered problem involves the fractional Caputo derivatives under some conditions on the ordered. As we combine Euler's method with composite Simpsons have been used to determine the solutions of these equations. We extend these numerical techniques to find the best solutions. Extended difference Euler technique is used for this. The results show that the extended Euler method is more accurate in terms of absolute error. Illustrative examples are given to demonstrate the high precision and good performance of the new class.

Keywords: Euler method, exact solution, approximate solution, fuzzy parameter, Caputo Fractional.

دراسة عددية لحل المعادلات التفاضلية التكاملية الضبابية المتضمنة مشتقة كابوتو الكسرية

رؤى نجم عبدالله و روكان خاجي

قسم الرياضيات- كلية العلوم – جامعة ديالى

الخلاصة

في هذا البحث، قمنا بتوسيع بعض الطرق العددية لحل المعادلات التفاضلية التكاملية الضبابية التي تتضمن مشتقة كابوتو الكسرية حيث قمنا بدمج طريقة اويلر مع سيمبسون المركبة لحل هذه المعادلات. وبينت النتائج حل اكثر دقة من حيث الخطأ المطلق. و لتوضيح ذلك قمنا بحل بعض الأمثلة التطبيقية لإثبات الدقة العالية و الأداء الجيد لهذه الطريقة الجديدة.

الكلمات المفتاحية: طريقة اويلر ، الحل المضبوط ، الحل التقريبي ، معاملات ضبابية ، كابوتو الكسوري.

Introduction

The fuzzy theory and integro-differential equations of fractional order are of great importance since they can be used in analyzing and modeling real world phenomena. The fuzzy fractional integro- differential equations have been recently used as effective tools in the modeling of many phenomena in various fields of applied sciences and engineering such as acoustic control, signal processing, electroencephalogram classification EEG, electromagnetics, optics, medicine, economics, and statistical physics and other scieces**.**

Since Zadeh in 1972 [6], both types fuzzy differential equations and integro differential equations have been studied extensively. The fuzzy derivative and its generalizations was introduced [1]. On the other hand, the fuzzy integral was introduced [2], they showed that fuzzy differential equation is the following form:

$$\begin{cases} z'(t,r) = h\big(t, z(t,r)\big) \\ \quad z(t_0, r) = z_0 \end{cases} \tag{1}$$

Has a unique solution in fuzzy case under the condition g satisfy the Lipschitz. Fuzzy Cauchy problem was studied [3],[4] investigated existence and uniqueness of solutions for fuzzy integro- differential equations with fuzzy kernel function[5],[11] proposed the extended of trapezoidal method to solve the fuzzy problem that has first order.[18] used fixed point theory to achieve the existence and uniqueness of Caputo -fractional fuzzy Volterra Fredholm integro-differential equation. The two-dimensional Legendre wavelet was studied [19]. They approximated the solution of Caputo-fractional fuzzy integro-differential equation.

The study of fuzzy fractional integro -differential equations has been introduced as a new branch of fuzzy theory. The analytical methods for finding the exact solutions of fuzzy integro-differential equations is very difficult, so the numerical technique is the best way to resort to it. The aims of this study to improve the accuracy of the numerical solutions of fuzzy fractional integro-differential equations. The Euler method has been to be able to solve these equations, but current practice has less accuracy with error in approximating the solution for large step size. We proposed extended Euler technique to solve fuzzy fractional integro-differential equations numerically. The results are expected to be more accurate as compared to be existing method. The contributions of this paper as follows: we derive an efficient method for computing the approximate solutions of the proposed model, and discover some properties which related between fuzzy theory and integro-differential equations. Also, we show that the control parameters contributes effectively to determination of approximate solutions for fuzzy fractional equations.

The paper is organized as follows: section 1. contains the Preliminaries. In section 2. methodology description for solving fuzzy integro-differential equations is given. In section 3, one example is presented. The conclusion of this paper is shown in Section 4.

1. Preliminaries

In this paper, we use the following notations: $X(t_n)$ and X_n are exact solution and approximate solution respectively in time t_n.

Definition (1.1) [12]: A fuzzy number $\boldsymbol{\nu}$ is a fuzzy subset of a real line which it satisfies the following conditions Convexity, normality and the membership of bounded support is upper semi continuous.

Any fuzzy number $\boldsymbol{\nu}$ can be represent by the following parametric forms $\left(\underline{\mathrm{v}}(r), \overline{\mathrm{v}}(r)\right), 0 \leq r \leq 1$. That satisfies

a) $\underline{\mathrm{v}}(r)$ is non-decreasing and bounded left over $0 \leq r \leq 1$

b) $\overline{\mathrm{v}}(r)$ is a bounded left continuous and non-increasing over $0 \leq r \leq 1$

For each r∈ $[0,1]$ then $\underline{\mathrm{v}}(r) \leq \overline{\mathrm{v}}(r)$.

Definition (1.2) [7]: The r-level set is defined as $[u]^r = \{s; u(s) \geq r\}, \ 0 \leq r \leq 1$

Consequently, $[u]^r$ can be written as close interval

$$[u]^r = \left[\underline{u}\,(r), \overline{u}(r)\right]$$

Definition (1.3) [10]: A triangular fuzzy number is a fuzzy set V in X that is characterized by a tri-ordered (a_l, a_c, a_r) in the space R^3 with $a_l \leq a_c \leq a_r$ such that $[V]^0 = [a_l, a_r]$ and $[V]^1 = \{a_c\}$. The r-level set of a triangular fuzzy number V is given by :
$[V]^r = [a_c - (1-r)(a_c - a_l), a_c + (1-r)(a_r - a_c)]$.

Proposition (1.4) [8]: Let $\mathcal{P}: [a,b] \times [0,1] \rightarrow X$ be a fuzzy function such that $\mathcal{P}(t,r) = \left(\underline{\mathcal{P}}(t,r), \overline{\mathcal{P}}(t,r)\right)$, then, If $\mathcal{P}$ is differentiable then $\underline{\mathcal{P}}(t,r)$ *and* $\overline{\mathcal{P}}(t,r)$ are differentiable functions and $\mathcal{P}'(t,r) = \left(\underline{\mathcal{P}'}(t,r), \overline{\mathcal{P}}'(t,r)\right)$

Definition (1.5) [9]: Let $\varphi: [a,b] \rightarrow X$. Then for any partition $\mathcal{P} = \{a = t_0, t_1, t_2, \dots, t_m = b\}$ and $\xi_i \in [t_i, t_{i+1}]$, $i = 0,1,2,\dots,m$ the definite integral of φ over $[a,b]$ is

$$\int_a^b \varphi(t)dt = \lim_{\vartheta \to 0} \mathcal{M}_{\mathcal{P}}$$

Where, $\vartheta = \max\{|t_{i+1} - t_i|, i = 0,1,2,\dots,m\}$ and $\mathcal{M}_{\mathcal{P}} = \sum_{i=1}^{m} \varphi(\xi_i)(t_{i+1} - t_i)$

In the case φ is a fuzzy and continuous function then for each fuzzy parameter $0 \leq r \leq 1$, its definite integral exists and [7]

$$\begin{cases} \underline{\left(\int_a^b \varphi(t,r)dt\right)} = \int_a^b \underline{\varphi}(t,r)dt \\ \overline{\left(\int_a^b \varphi(t,r)dt\right)} = \int_a^b \overline{\varphi}(t,r)dt \end{cases} \tag{2}$$

Definition (1.6) [10]: Let $x = \left(\underline{x}(r), \overline{x}(r)\right)$ and $y = \left(\underline{y}(r), \overline{y}(r)\right), 0 \leq r \leq 1$ be fuzzy numbers. The distance between them is defined as follows:

$$d(x,y) = \left[\int_0^1 \left(\underline{x}(r) - \underline{y}(r)\right)^2 dr + \int_0^1 \left(\overline{x}(r) - \overline{y}(r)\right)^2 dr\right]^{0.5} \tag{3}$$

Definition (1.7) [13]: The α - Caputo fractional derivative of a real $h \in C(a,b)$, is

$$D_C^{\alpha} h(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n \in N \\ h^{(n)}(t), & \alpha = n \in N \end{cases} \quad (4)$$

Definition (1.8) [13]: The α - Caputo fractional integral of a real function $h \in C(a,b)$ is

$$I_C^{\alpha} h(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(\tau)}{(t-\tau)^{1-\alpha}} d\tau, & 0 < \alpha \\ h(t), & \alpha = 0 \end{cases} \quad (5)$$

Proposition (1.9)[15]: Assume that the Laplace transform of non-negative function φ exists on $[a, \infty)$ with some $a \in R$. Let $\alpha > 0$ and $n = [\alpha]$. Then, for $s > \max\{0, a\}$ we have

$$\mathcal{L} I_0^{\alpha} \varphi(s) = \frac{1}{s^{\alpha}} \mathcal{L}\varphi(s) \quad (6)$$

$$\mathcal{L} D_0^{\alpha} \varphi(s) = s^{\alpha} \mathcal{L}\varphi(s) - \sum_{k=1}^{n} s^{\alpha-k} \varphi^{(k-1)}(0) \quad (7)$$

Definition (1.10) [14]: The Mittag-Leffler function $E_{\alpha,\beta}(\tau)$ for any $\alpha, \beta > 0$ is

$$E_{\alpha,\beta}(\tau) = \sum_{n=1}^{\infty} \frac{\tau^n}{\Gamma(m\alpha+\beta)} \quad (8)$$

Proposition (1.11)[14]: For each values $\alpha, t, \eta > 0$, the following statements are hold, $\lambda \in C$

If $y(t) = E_{\alpha}(-\eta\, t^{\alpha})$ then $\mathcal{L}y(t) = \lambda^{\alpha-1}(\lambda^{\alpha} - \eta)^{-1}$ (9)

$$\frac{d}{dr} E_{\alpha}(-\eta\, r^{\alpha}) = \sum_{m=1}^{\infty} \frac{(-\eta)^m r^{\alpha m-1}}{\Gamma(m\alpha)}$$

Proposition (1.12)[17]: The following relation is hold for any $\alpha, \beta > 0$ and $\lambda \in C$

If $y(t) = t^{\alpha-1} E_{\beta,\alpha}(at^{\beta})$ then $\mathcal{L}y(t) = \lambda^{\beta-\alpha}(\lambda^{\beta} - a)^{-1}$ (10)

Definition (1.13) [16]: let X(t) be a α - fractional differentiable function such that $0 < \alpha \leq 1$. Then, fractional Euler method for $X(t)$ at t_n, can be written as:

$$X_n = X_{n-1} + \mathrm{D}_{\mathrm{C}}^{\alpha} X(t_{n-1}) \frac{h^{\alpha}\Delta_{n-1}}{\Gamma(\alpha+1)} \tag{11}$$

Where, $\Delta_{n-1} = (n)^{\alpha} - (n-1)^{\alpha}$

2. Methodology Description

The fuzzy fractional integro-differential equation is

$$\begin{cases} \mathrm{D}_{\mathrm{C}}^{\alpha} X(t,r) + P(t,r)\, X(t,r) = f(t,r) + \beta \int_a^b k(t,s) X(s,r) ds \\ X(a) = X_0(r) \end{cases} \tag{12}$$

Where, $\mathrm{D}_{\mathrm{C}}^{\alpha}$ is a Caputo fractional derivative of order $0 < \alpha \le 1$ which defined on $[a,b]$ and is already given, $\beta > 0$, $r \in [0,1]$ is a fuzzy parameter, $k(t,s)$ over $s,t \in [a,b]$ is the kernel of this equation.

In parametric form, equation (12) is represented as follows:

$$\begin{cases} \mathrm{D}_{\mathrm{C}}^{\alpha} \underline{X}(t,r) + \underline{P(t,r)\, X(t,r)} = \underline{f}(t,r) + \beta \int_a^b \underline{k(t,s) X(s,r)} ds \\ \mathrm{D}_{\mathrm{C}}^{\alpha} \overline{X}(t,r) + \overline{P(t,r)\, X(t,r)} = \overline{f}(t,r) + \beta \int_a^b \overline{k(t,s) X(s,r)} ds \\ \underline{X}(a) = \underline{X}_0(r) \\ \overline{X}(b) = \overline{X}_0(r) \end{cases} \tag{13}$$

In addition, $\underline{P(t,r)\, X(t,r)} = \underline{P}(t,r)\underline{X}(t,r)$, $\overline{P(t,r)\, X(t,r)} = \overline{P}(t,r)\overline{X}(t,r)$, $P(t,r) = \left(\underline{P}(t,r), \overline{P}(t,r)\right)$, $\underline{k(t,s)X(s,r)} = k(t,s)\underline{X}(s,r)$, $\overline{k(t,s)X(s,r)} = k(t,s)\overline{X}(s,r)$

The formula of Euler method is:

$$\begin{cases} X_n = X_{n-1} + \mathrm{D}_{\mathrm{C}}^{\alpha} X(t_{n-1}) \frac{h^{\alpha}\Delta_{n-1}}{\Gamma(\alpha+1)} \\ X_{n-1} = X_n - \mathrm{D}_{\mathrm{C}}^{\alpha} X(t_n) \frac{h^{\alpha}\Delta_{n-1}}{\Gamma(\alpha+1)} \end{cases} \tag{14}$$

Where Γ is a gamma function. Equations in (14) give the following formula:

$$X_n = X_{n-1} + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)} \left(\mathrm{D}_{\mathrm{C}}^{\alpha} X(t_{n-1}) + \mathrm{D}_{\mathrm{C}}^{\alpha} X(t_n) \right) \tag{15}$$

To use the following notations in the equations (13)

$$\underline{\psi}\left(t,r,X(t,r),\int_a^b \underline{k(t,s)X(s,r)}ds\right) = -\,\underline{P(t,r)\,X(t,r)} + \underline{f}(t,r) + \beta\int_a^b \underline{k(t,s)X(s,r)}ds$$

$$\overline{\psi}\left(t,r,X(t,r),\int_a^b \overline{k(t,s)X(s,r)}ds\right) = -\,\overline{P(t,r)\,X(t,r)} + \overline{f}(t,r) + \beta\int_a^b \overline{k(t,s)X(s,r)}ds$$

$$\begin{cases} \underline{\psi_n} = -\,\underline{P_n\,X_n} + \underline{f_n} + \beta\int_a^b \underline{k(t_n,s)X_n}ds \\ \overline{\psi_n} = -\,\overline{P_n\,X_n} + \overline{f_n} + \beta\int_a^b \overline{k(t_n,s)X_n}ds \end{cases} \tag{16}$$

Where, $\underline{P_n\,X_n} = \underline{P(t_n,r)\,X(t_n,r)}$, $\underline{f_n} = \underline{f}(t_n,r)$. $\underline{k(t_n,s)X_n} = k(t_n,s)\,\underline{X(t_n,r)}$, $\overline{P_n\,X_n} = \overline{P(t_n,r)\,X(t_n,r)}$, $\overline{f_n} = \overline{f}(t_n,r)$ and $\overline{k(t_n,s)X_n} = k(t_n,s)\overline{X(t_n,r)}$

Now, applying these notations and the formula in (15) on equations in (13), we have

$$\begin{cases} \underline{X_n} = \underline{X_{n-1}} + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left(\underline{\psi_{n-1}} + \underline{\psi_n}\right) \\ \overline{X_n} = \overline{X_{n-1}} + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left(\overline{\psi_{n-1}} + \overline{\psi_n}\right) \end{cases} \tag{17}$$

Now, using composite Simpsons on with n subintervals and s belong to $[a,b]$, the integral part of equations in (16) is approximated by

$$\underline{I}_0 = \frac{2h}{3}\left(k(t_0,t_0)\underline{X_0}\right)$$

$$\overline{I}_0 = \frac{2h}{3}\left(k(t_0,t_0)\overline{X_0}\right)$$

$$\underline{I}_1 = \frac{h}{3}\left(k(t_1,t_0)\underline{X_0} + k(t_1,t_1)\underline{X_1}\right)$$

$$\overline{I}_1 = \frac{h}{3}\left(k(t_1,t_0)\overline{X_0} + k(t_1,t_1)\overline{X_1}\right)$$

$$\begin{cases}\underline{I}_n=\int_a^b \underline{k(t_n,s)X_n}ds=\frac{h}{3}\left(k(t_n,t_0)\underline{X_0}+4\sum_{k=1}^{n-1}k(t_n,t_k)\underline{X_k}+k(t_n,t_n)\underline{X_n}\right)\\ \overline{I}_n=\int_a^b \overline{k(t_n,s)X_n}ds=\frac{h}{3}\left(k(t_n,t_0)\overline{X_0}+4\sum_{k=1}^{n-1}k(t_n,t_k)\overline{X_k}+k(t_n,t_n)\overline{X_n}\right)\end{cases} \quad (18)$$

Consequently, the equations in (16) become

$$\begin{cases}\underline{\psi_n}=-\underline{P_n\,X_n}+\underline{f_n}+\beta\underline{I}_n\\ \overline{\psi_n}=-\overline{P_n\,X_n}+\overline{f_n}+\beta\overline{I}_n\end{cases} \quad (19)$$

By substituting equations (19) and (18) in equations (17), we get on the following formulas n=2,3,

$$\underline{X_n}=\left\{1+\frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\underline{P_n}-\frac{h^{\alpha+1}\Delta_{n-1}}{6\Gamma(\alpha+1)}\beta k(t_n,t_n)\right\}^{-1}\left\{\underline{X_{n-1}}+\frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left(\underline{\psi_{n-1}}\right)+\frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left\{\underline{f_n}+\frac{\beta h}{3}\left(k(t_n,t_0)\underline{X_0}+4\sum_{k=1}^{n-1}k(t_n,t_k)\underline{X_k}\right)\right\}\right\}$$

$$\overline{X_n}=\left\{1+\frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\overline{P_n}-\frac{h^{\alpha+1}\Delta_{n-1}}{6\Gamma(\alpha+1)}\beta k(t_n,t_n)\right\}^{-1}\left\{\overline{X_{n-1}}+\frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left(\overline{\psi_{n-1}}\right)+\frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left\{\overline{f_n}+\frac{\beta h}{3}\left(k(t_n,t_0)\overline{X_0}+4\sum_{k=1}^{n-1}k(t_n,t_k)\overline{X_k}\right)\right\}\right\}$$

(20)

The first and second states are expressed by

$$\underline{X_1}=\left\{1+\frac{h^{\alpha}}{2\Gamma(\alpha+1)}\underline{P_1}-\frac{h^{\alpha+1}}{6\Gamma(\alpha+1)}\beta k(t_1,t_1)\right\}^{-1}\left\{\underline{X_0}+\frac{h^{\alpha}}{2\Gamma(\alpha+1)}\left(\underline{\psi_0}\right)+\frac{h^{\alpha}}{2\Gamma(\alpha+1)}\left\{\underline{f_1}+\frac{\beta h}{3}\left(k(t_1,t_0)\underline{X_0}\right)\right\}\right\}$$

$$\overline{X_1} = \left\{1 + \frac{h^{\alpha}}{2\Gamma(\alpha+1)}\overline{P_1} - \frac{h^{\alpha+1}}{6\Gamma(\alpha+1)}\beta k(t_1,t_1)\right\}^{-1}\left\{\overline{X_0} + \frac{h^{\alpha}}{2\Gamma(\alpha+1)}\left(\overline{\psi_0}\right) + \frac{h^{\alpha}}{2\Gamma(\alpha+1)}\left\{\overline{f_1} + \frac{\beta h}{3}\left(k(t_1,t_0)\overline{X_0}\right)\right\}\right\}$$

(21)

$$\underline{X_2} = \left\{1 + \frac{h^{\alpha}(2^{\alpha}-1)}{2\Gamma(\alpha+1)}\underline{P_2} - \frac{h^{\alpha+1}(2^{\alpha}-1)}{6\Gamma(\alpha+1)}\beta k(t_2,t_2)\right\}^{-1}\left\{\underline{X_1} + \frac{h^{\alpha}(2^{\alpha}-1)}{2\Gamma(\alpha+1)}\left(\underline{\psi_1}\right) + \frac{h^{\alpha}(2^{\alpha}-1)}{2\Gamma(\alpha+1)}\left\{\underline{f_2} + \frac{\beta h}{3}\left(k(t_2,t_0)\underline{X_0} + 4k(t_2,t_1)\underline{X_1}\right)\right\}\right\}$$

$$\overline{X_2} = \left\{1 + \frac{h^{\alpha}(2^{\alpha}-1)}{2\Gamma(\alpha+1)}\overline{P_2} - \frac{h^{\alpha+1}(2^{\alpha}-1)}{6\Gamma(\alpha+1)}\beta k(t_2,t_2)\right\}^{-1}\left\{\overline{X_1} + \frac{h^{\alpha}(2^{\alpha}-1)}{2\Gamma(\alpha+1)}\left(\overline{\psi_1}\right) + \frac{h^{\alpha}(2^{\alpha}-1)}{2\Gamma(\alpha+1)}\left\{\overline{f_2} + \frac{\beta h}{3}\left(k(t_2,t_0)\overline{X_0} + 4k(t_2,t_1)\overline{X_1}\right)\right\}\right\}$$

(22)

3. Illustrative example

To show the efficiency and accuracy of the proposed technique with various values of step size, we consider the following example.

Example: Consider the following fuzzy fractional integro-differential equation taken from (12)

$$\begin{cases} D_C^{\alpha}X(t,r) + X(t,r) = \left((3+3r)\sinh(t), (8-2r)\sinh(t)\right) + \int_0^1 (t-s)X(s,r)ds \\ X(0,r) = \left((3+3r),(8-2r)\right), \quad t \in [0,1], 0 \le r \le 1 \end{cases} \tag{23}$$

Where, $0 < \alpha \le 1$, the exact solution is given by

$$X(t,r) = \left((3+3r)\left(t^{\alpha+1}E_{2,\alpha+2}(t^2)+1\right), (8-2r)\left(t^{\alpha+1}E_{2,\alpha+2}(t^2)+1\right)\right) \tag{24}$$

The approximate solution by using extended difference Euler method is given by

$$\underline{X_n} = \left\{1 + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\right\}^{-1} \left\{\underline{X_{n-1}} + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left(\underline{\psi_{n-1}}\right) + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left\{(3+3r)\sinh(t_n) + \frac{h}{3}\left((t_n - t_0)\underline{X_0} + 4\sum_{k=1}^{n-1}(t_n - t_k)\underline{X_k}\right)\right\}\right\}$$

$$\overline{X_n} = \left\{1 + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\right\}^{-1} \left\{\overline{X_{n-1}} + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left(\overline{\psi_{n-1}}\right) + \frac{h^{\alpha}\Delta_{n-1}}{2\Gamma(\alpha+1)}\left\{(8-2r)\sinh(t_n) + \frac{h}{3}\left((t_n - t_0)\overline{X_0} + 4\sum_{k=1}^{n-1}(t_n - t_k)\overline{X_k}\right)\right\}\right\}$$

(25)

Where

$$\begin{cases} \underline{\psi_n} = -\underline{X_n} + (3+3r)\sinh(t_n) + \underline{I}_n \\ \overline{\psi_n} = -\overline{X_n} + (8-2r)\sinh(t_n) + \overline{I}_n \end{cases} \quad (26)$$

Approximate solutions $\underline{X_n}$,$\overline{X_n}$ can be found by solving equations in (25) (see Fig. 1., 2, 3, 4, 5, 6) And Table 1, 2, 3)

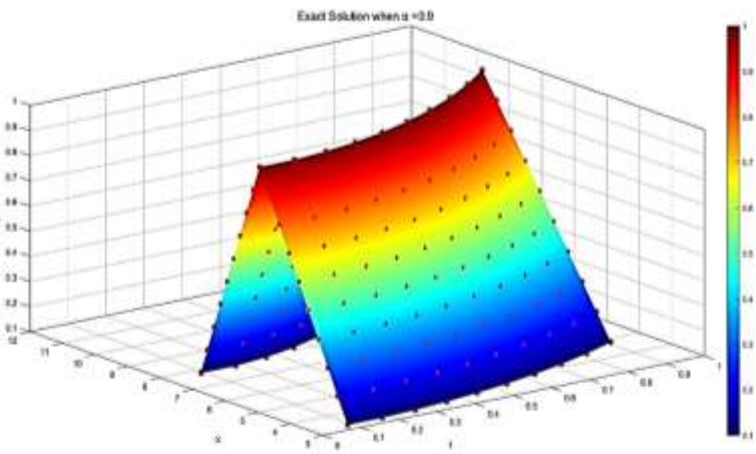

Figure 1: Exact Solution when α = 0.9

Table 1: $h = 0.1, \alpha = 0.9$

t	d
0	**0**
0.3	**0.0052**
0.5	**0.0149**
0.7	**0.0188**
0.9	**0.0237**

Academic Science Journal

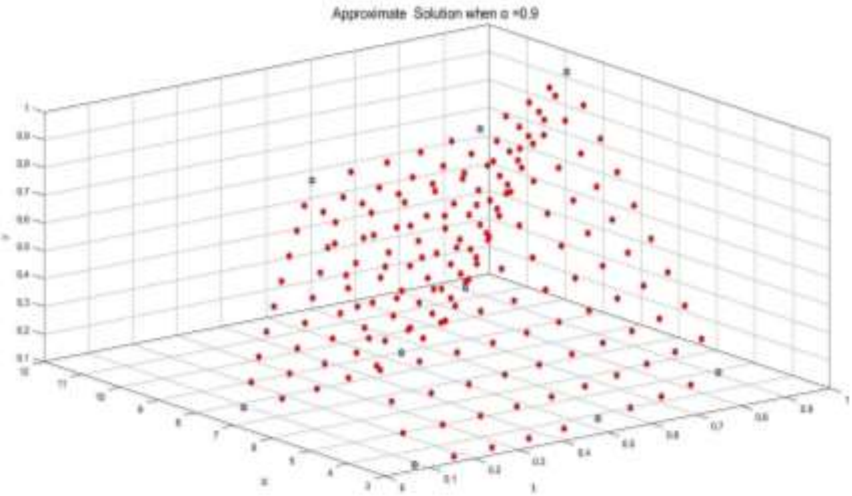

Figure 2: Approximate Solution when α =

Table 2 $h = 0.01$, $\alpha = 0.9$

t	D
0	**0**
0.3	**0.0015**
0.5	**0.0068**
0.7	**0.0145**
0.9	**0.0209**

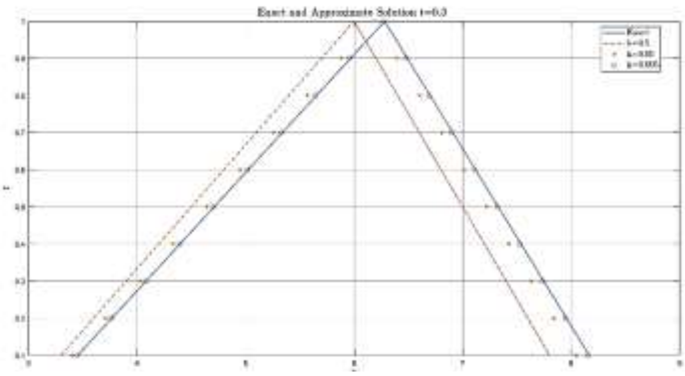

Figure 3: Exact and Approximate Solution at t=0.3 , $\alpha = 0.9$

Table 3 $h = 0.001$,, $\alpha = 0.9$

t	d
0	**0**
0.3	**0.0014**
0.5	**0.0067**
0.7	**0.0150**
0.9	**0.0214**

Conclusion

The extended difference Euler technique for solving fractional order fully fuzzy integro-differential Equations was considered. This technique proved it's efficient in solving of proposed equations by providing the best approximate of its solutions. We showed that the fractional parameter played a fundamental and important role in reducing the error rate which resulted from the approximation of solutions for fuzzy fractional integro-differential Equations. Thus, our work in this paper, can be extended to multivariate fuzzy fractional equations. Finally, we would like to refer that the proposed equation can be applied to various fields such as environmental, medicine, economy, engineering and biomedical.

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