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A Numerical Study of Linear Two-Points Boundary Value Problems with ODEs of Fourth-Order Using RKM Method

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Abstract

This study established the numerical RKM approach for solving linear two-point boundary value problems involving fourth-order ordinary differential equations. However, the proposed developed numerical RKM method has been tested using some implementations in order to compare it with the exact solutions to establish the method's validity. Furthermore, this comparison demonstrates that the proposed direct integrator is more efficient than the indirect method in terms of efficiency and accuracy. In addition, numerical implementations are used in order to show the efficiency and time-based complexity of function evaluations. This direct method's suggested strategy, which has wonderful qualities like quick and efficient calculation, also requires fewer computational employees.

Keyword: RKM, Ordinary Differential Equations; Boundary Value Problems, Order, DEs, ODEs.

Introduction

Fourth-order boundary-value problems (BVPs) arise in mathematical modeling of science and engineering, for example, cantilever beam deflection under concentrated load, beam deformation and plate deflection theory, obstacle problems, temperature distribution of the trapezoidal profile's radiation fin, and a variety of other engineering applications. However, the linear boundary value problems (LBVPs) with ordinary differential equations (ODEs) are used to represent a wide range of situations, including electrostatic potential between two concentric metals, chemical reactions, heat transmission, and bean deflection. A boundary value issue with four boundary conditions can be used to introduce these problems. Numerous authors have employed numerical and approximate techniques to address LBVPs, with specific methodologies outlined in references [1-10] which introduced the construction of a numerical algorithm for solving second-order LBVPs with Dirichlet and Neumann boundary conditions involves the utilization of Walsh wavelets and semi-orthogonal B-spline wavelets. Na, (1980) [1] achieved the numerical solution for second-, third-, and fourth-order BVPs by converting them into initial value problems (IVPs) and employing methods such as the nonlinear shooting method of reduced physical parameters and the method of invariant embedding. Our current approach is versatile, and applicable to both BVPs and IVPs with slight adjustments, eliminating the need for transforming BVPs into IVPs or vice versa. Two-point BVPs are found in all fields of engineering and science. The boundary conditions (BCs) are mentioned at two stages in these issues. To make matters worse, the governing differential equations for the most majority of such situations are nonlinear; because analytic solutions do not exist in general, solutions have to be sought through numerical approaches. There are two types of approaches for numerically solving such problems: iterative and non-iterative methods. Non iterative solutions can always be obtained for LBVPs. Iteration is generally required for nonlinear situations. It should be noted, however, that there are numerous strategies for eliminating iteration of the answer, resulting in significant reductions in computing time [1]. However, in mechanics, the technique was used to analyze many two- and multi-BVPs. In the last years, many numerical techniques for solving LBVPs have been studied, for example, the numerical method utilizing Walsh wavelet packet bases as trial functions within the formulation of the least square method where Walsh-wavelet packet basis functions enable the formulation of an efficient numerical approach for analyzing multi-point linear BVPs [3].

A method based on Non-polynomial spline functions of quadratic nature is devised to obtain approximation solutions to a set of second-order equations BVPs including obstruction, unilateral, and contact problems. The present approach incurs lower computational costs and yields more accurate approximations compared to alternative collocation, finite-difference, and spline methodologies. The method's convergence analysis is covered. An illustrative numerical instance is provided to demonstrate the new method's practical application [4]. A novel numerical approach for computing estimations of the solution to a collection of boundary-value problems of third order related to obstruction, unilateral, and contact problems is developed using quartic non-polynomial splines. It is demonstrated that the novel method produces superior approximations than other collocation, spline methods, and finite-difference. The method's convergence analysis is presented using standard methodologies. A numerical illustration is provided to demonstrate the applicability. A computational technique is formulated for approximating the solution of a set of second-order boundary value problems (BVPs) by employing non-polynomial spline functions equivalent to cubic splines, which proved that the proposed method creates superior approximations compared to alternative collocation, finite difference, and spline techniques. The convergence analysis of this method is discussed. A numerical example is provided to illustrate the method's practical use studied the shooting techniques to approximate the solution of Troesch’s two-point BVPs [5]. A method for solving multipoint BVPs is described. This method can be used to solve the generic form of multipoint BVPs. Numerical results are presented to demonstrate the effectiveness of the devised method [8]. For second-order, non-linear BVPs, finite-difference algorithms of orders six and eight are described. Both techniques are cost-effective because they perform few function evaluations at inside grid points. The methods are simple to put into practice. The methods' convergence is addressed. Numerical examples are used to highlight the computational order of convergence. Non-polynomial splines are utilized to construct a class of numerical algorithms for computing approximations to the solution of sixth-order BVPs with two-point boundary conditions. The conventional approach yields second-, fourth- , and sixth-order convergence. It is demonstrated that the current methods yield better estimates than other spline and domain decomposition methods [11].

The objective of this article is to develop a method for solving the linear fourth-order BVPs numerically which arising in the mathematical modeling of different engineering applications. This paper is concerned with the solution of LBVPs involving fourth-order ODEs, which are as follows:

$$y^{(4)}(x) = P(x)y'''(x) + Q(x)y''(x) + R(x)y'(x) + S(x)y(x) + T(x), \quad (1)$$

Where $P(x) > 0,\ \ a \le x \le b$,

With the boundary conditions

$$y(a) = \alpha_0,\ \ y'(a) = \alpha_1,\ y''(a) = \alpha_2,$$

$$y(b) = \beta, \quad (2)$$

where a , b , α , β, γ and δ are the given constants.

Furthermore, to assess the suitability of the devised approach, our emphasis is placed on the LBVPs of the type in Equation (1) with the boundary conditions in Eq. (2).

The following structure is used to organize the paper. Section 2. The preliminary introduces some fundamental concepts connected to the subject under consideration in this work. Section 3 presents a generic formulation of the numerical strategy based on the RKM method for solving 4^{th}-order ODEs. Furthermore, in Section 4, the major conclusions of the suggested method of solutions, as well as a summary of the analysis and several numerical examples, are presented. Finally, numerical results are shown in Section 4 to demonstrate the efficiency of the proposed direct method with four tested problems.

1. Preliminary

This section introduced some basic concepts related to the problem of study in this paper.

- **Quasi Linear ODEs of 4^{th} Order**

The special quasi-linear ODE of fourth order has the following form

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$$y^{(4)}(x) = f(x, y(x)),\ x \geq a \tag{3}$$

With the initial conditions (I.Cs.)

$$y^{(i)}(a) = y_0^i,\ i = 0, 1, 2, 3. \tag{4}$$

with the special properties for the function f does not depend directly on the derivatives $(y^{(\eta)}(x))$, $\eta = 1,2,3,4$. Where $f: \mathcal{R} \times \mathcal{R}^d \to \mathcal{R}^d$.

A lot of authors derived numerical methods of one-step, multistep or embedded types for solving different orders ODEs [11-25]. Historically, scientists and engineers used to solve ODEs of the fourth order by converting ODE into a system of ODEs of the first order and then numerically solving this system. This strategy was wasteful of both digital and human resources. Numerous authors derived direct explicit one step methods for solving different orders of ODEs [1-11] while the others derived direct embedded or multistep methods for solving different orders of ODEs [12-15] provided these, direct numerical approaches would be the most effective method for solving ODEs of orders three to ten. However, Mechee and Kadhim derived a direct RKM method for solving ODE of fourth order in Equation (2) with the initial conditions in Equation (3) [14].

- **RKM Method for Solving 4^{th} –Order ODEs**

To solve the class of 4^{th} -order ODEs in Equation (2) with the initial conditions in Equation (3), the RKM technique with a ş-stage can be found in [4] which derived the order conditions of the proposed RKM method and then, they evaluated the parameters of RKM integrator $a_{ij}, c_i, b_i^{(ii)}$ for $i, j = 1,2, \ldots, s$, $ii = 0,1, \ldots, 3$.

- **The Direct Numerical RKM Method [14]**

To solve the ODE of fourth-order in Equation (3) with the initial conditions in Equation (4), we use the general form of the RKM method with an s-stage in the following Equations (5)-(10):

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y_n^{(3)} + h^4 \sum_{i=1}^{s} b_i^{(0)} K_i,$$

$$y'_{n+1} = y'_n + h y''_n + \frac{h^2}{2} y_n^{(3)} + h^3 \sum_{i=1}^{s} b_i^{(1)} K_i, \qquad (6)$$

$$y''_{n+1} = y''_n + h y_n^{(3)} + h^2 \sum_{i=1}^{s} b_i^{(2)} K_i \qquad (7)$$

$$y_{n+1}^{(3)} = y_n^{(3)} + h \sum_{i=1}^{s} b_i^{(3)} K_i, \qquad (8)$$

Where

$$K_1 = f(x_n, y_n), \qquad (9)$$

$$K_i = f\left(x_n + c_i h, y_n + h c_i y'_n + \frac{h^2}{2} c_i^2 y''_n + \frac{h^3}{6} c_i^3 y_n^{(3)} + h^4 \sum_{j=1}^{i-1} a_{ij} K_j\right)$$

For $i = 2,3, \dots, s$. Where h is the norm of subintervals RKM method.

The explicit RKM method has the following real values for the parameters $c_i, a_{ij}, b_i^{(0)}, b_i^{(1)}, b_i^{(2)}, b_i^{(3)}$ for $i, j = 1,2, \dots, s$. The following table of coefficients can be used to describe the RKM Method in Butcher notation: (see Table 1)

Table 1: Table of RKM Method

C	A
	b^T b'^T b''^T $b^{(3)T}$

The authors in [14] derived two RKM methods of three- and fourth-stages respectively as in the following Table 2 and Table 3.

Table 2: Table of RKM Method of three-stages

0	0		
$\frac{5}{6}$	$\frac{1}{2}$	0	
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	0
	$\frac{1}{60}$	$\frac{1}{3240}$	$\frac{1}{81}$
	$\frac{1}{20}$	$\frac{1}{180}$	$\frac{1}{9}$
	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{3}$
	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{2}$

Table 3: Table of RKM Method fourth-stages

0	0			
$\frac{1}{2}$	$\frac{3}{160}-\frac{\sqrt{15}}{240}$	0		
$\frac{1}{2}-\frac{\sqrt{15}}{10}$	$\frac{-1}{2}$	$\frac{1}{2}$	0	0
$\frac{1}{2}+\frac{\sqrt{15}}{10}$	$\frac{-51}{100}-\frac{22\sqrt{15}}{75}$	$\frac{1}{100}+\frac{\sqrt{15}}{5}$	$\frac{1}{2}$	0
	0	$\frac{1}{108}$	$\frac{7}{432}+\frac{\sqrt{15}}{240}$	$\frac{7}{432}-\frac{\sqrt{15}}{240}$
	0	$\frac{1}{18}$	$\frac{1}{18}+\frac{\sqrt{15}}{72}$	$\frac{1}{18}-\frac{\sqrt{15}}{72}$
	0	$\frac{2}{9}$	$\frac{5+\sqrt{15}}{36}$	$\frac{5-\sqrt{1}}{36}$
	0	$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$

Results

In this section, we introduce the main results of the method.

Method of Solutions

In general, the shooting technique for solving LBVPs of ODEs is based on the placement of this problem by two IVPs in which have the solutions y_1(x) and y_2(x). There are numerous methods for approximating these solutions y_1(x) and y_2(x), and once these approximations are known, the solution to the boundary-value issue is approximated as a linear combination of y_1(x) and y_2(x), The procedure for finding the solutions is depicted graphically in Figure 1. The linear shooting method is generalized for solving the boundary value problem of the fourth order which is described in Equation (1) with the boundary conditions in Equation (2) and is based on the replacement of the LBVPs in Equation (1) with the boundary conditions in Equation (2) by two (IVPs) which have the solutions y_1(x) and y_2(x) which shown as follows:

$$y_1^{(4)}(x) = P(x)\, y_1'''(x) + Q(x)\, y_1''(x) + R(x)\, y_1'(x) + S(x)\, y_1(x) + T(x), \quad (11)$$

Where $P(x) > 0,\ \text{a} \le x \le \text{b}$

With the boundary condition

$$y_1^{(i)}(\text{a}) = \alpha_i,;\, i = 0,1,2;\ \ y_1'''(\text{a}) = 1. \quad (12)$$

and

$$y_2^{(4)}(x) = P(x)\, y_2'''(x) + Q(x)\, y_2''(x) + R(x)\, y_2'(x) + S(x)\, y_2(x), \quad (13)$$

$y_2{}^{(i)}(\mathrm{a}) = 0 \; for \; i =$
0,1,2,3
(14)

Where

$y(x) = y_1(x) +$
$\theta y_2(x),$

and $\theta = \frac{\beta - y_1(\mathrm{b})}{y_2(\mathrm{b})}$.

Then, the solution of BVP in Equations (1)-(2) is a linear combination of the solutions of two IVPs (11)-(12) and (13)-(14) as written in Equation (15)

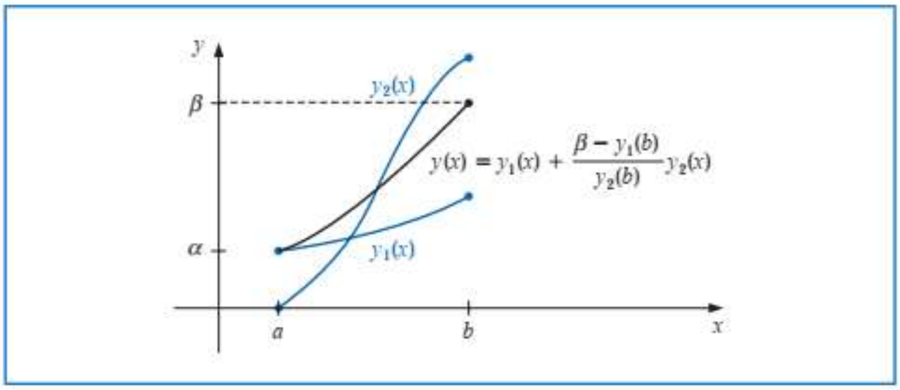

Figure 1: Combination Solutions of BVP Using Shooting Method where $\alpha = \alpha_0$

Numerical Results

In this section, we demonstrate the effectiveness of the direct method with four tested problems.

Example 1

Consider $\frac{d^4 y(x)}{dx^4} = 16y(x), \;\; 0 \le x \le \pi,$

with the boundary conditions $y(0) = y(\pi) = y''(0) = 0, y'(0) = 2$.

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Exact solution: $y(x) = \sin(2x)$.

Example 2

Consider $\frac{d^4 y(x)}{dx^4} = y(x) + 15e^{-2x}, \; 0 \leq x \leq 1,$

With the boundary conditions $y(0) = 1, \; y(1) = e^{-2}, y'(0) = -2, y''(0) = 4.$ Exact solution: $y(x) = e^{-2x}$.

Example 3

Consider $\frac{d^4 y(x)}{dx^4} = y(x) - x^7 + 840\, x^3 \;, \; 0 \leq x \leq 1$

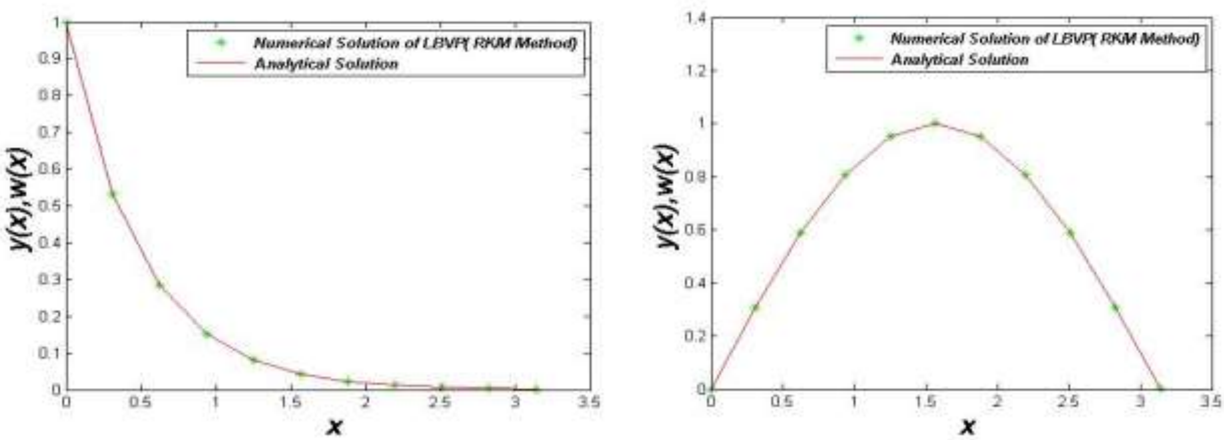
With the boundary conditions $y^{(k)}(0) = 0, \; k = 0,1,2, y(1) = 1.$ Exact solution: $y(x) = x^7$.

Example 4

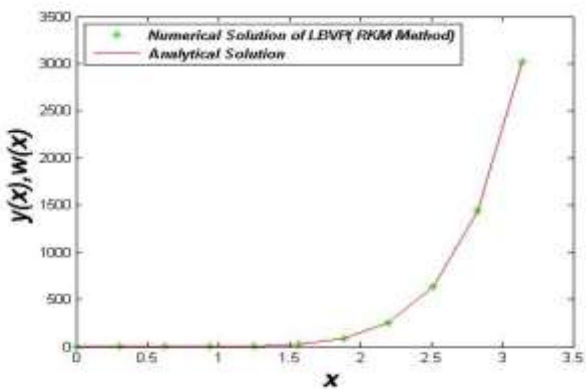
Consider $\frac{d^4 y(x)}{dx^4} = y(x) - \frac{1}{x}\frac{24}{x^5}. \;, \; 1 \leq x \leq 2,$

With the boundary conditions $y(1) = 1, y(2) = \frac{1}{2}, y'(1) = -1, y''(1) = 2.$

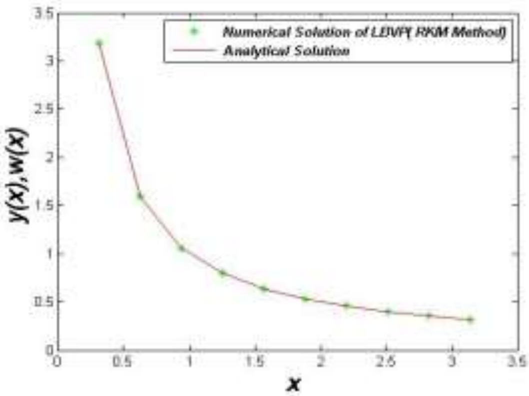
Exact solution: $y(x) = \frac{1}{x}$.



(a) (b)



(c)



(d)

Figure 2: A Comparison of Numerical Solutions with The Exact Solutions for Examples (a) 1, (b) 2, (c) 3, and, (d) 4**.**

Discussions and Conclusion

In this study, we developed a direct numerical method for solving special BVPs of fourth-order, quasi-linear ODEs. The improved RKM integrators for solving ODEs of the order fourth-order is a novel aspect of this work. The purpose of this study is to develop an explicit direct integrator for a particular class of fourth-order ODEs. We have examined the effectiveness of the proposed RKM method using a variety of quasi-linear, fourth-order ODEs examples. The numerical results of the ODEs in Figure 1 demonstrated that the proposed method yielded analytical solutions that were identical. However, based on the numerical outcomes produced by the RKM method, we can infer that RKM is more accurate and effective than the current method. Finally, the constructed RKM method: is more cost-effective in terms of computational-time, than existing methods.

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