



Z-Small Retractable Modules

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ABSTRACT

Let C be the left unitary R -module and let R be a commutative ring with 1. In this paper, we introduced and studied the concept of Z -small retractable module where a R -module C is said to be Z -small retractable (Z.S.R.) if $\text{Hom}(C, K) \neq 0$, for every nonzero Z -small submodule K of C . Equivalently, C is Z -small retractable (Z.S.R.) if there exists a homo-morphism $f: C \rightarrow N$ whenever $0 \neq N \ll_Z C$ as an extension of a retractable module (where an R -module C is called retractable if $\text{Hom}_R(C, N) \neq 0$ for each non-zero submodule N of C) and we give some of their advantages characterizations, examples and properties are studied and many relationships between these classes and other related concepts are presented and studied.

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1. INTRODUCTION

Let R be a commutative ring with identity 1 and C be left unitary R -module. In [1], Amina and Alaa introduced and studied the concept of Z -small submodules, where a submodule N of an R -module C is called Z -small submodule $N \ll_Z C$ if $N + B = C$ with $B \supseteq Z_2(C)$ and $B \leq M$, then $B = C$ (where the second singular submodule $Z_2(C)$ of C is the submodule of C containing $Z(C)$ such that $\frac{Z_2(C)}{Z(C)}$ is the singular submodule of $\frac{C}{Z(C)}$ [2]). An R -module C is said to be Z -small compressible if C can be embedded in every of it is nonzero Z -small submodule of C . Equivalently, C is Z -small compressible if there exists a mono-morphism $f: C \rightarrow N$ whenever $0 \neq N \ll_Z C$ [3]. Recall that C is called Z -small quasi-Dedekind modules if for all $0 \neq f \in \text{End}_R(C)$ implies that $\ker f \ll_Z C$ [4]. An R -module C is called retractable if $\text{Hom}_R(C, N) \neq 0$ for each non-zero submodule N of C , [5].

In this paper, we introduce and study the concept of Z -small retractable as a generalization of retractable modules. We present several properties, characterization, and examples. In addition, we show that under conditions. Z -small retractable and Z -small compressible are equivalent. Some of their advantages, characterizations and examples are given. We also study the relation between the Z -small retractable module, Z -small quasi-Dedekind modules, and some classes of modules.

2. Z-Small Retractable Modules

In this section, we present the concept of Z -small retractable modules as a generalization of retractable modules. We provide some basic properties, examples, and characterizations of this concept.

Definition (2.1):

An R -module C is said to be Z -small retractable ($Z.S.R.$) if $om(C, K) \neq 0$, for every non-zero Z -small submodule K of C .

That means, C is ($Z.S.R.$) if there exists a homo-morphism $f: C \rightarrow N$, for every non-zero Z -small submodule N of C .

Remarks and Examples (2.2):

1. It's obvious that every Z -small compressible module is a ($Z.S.R.$) module.
2. The converse of (1) is not true: For example, Z_4 is ($Z.S.R.$) but not Z -small compressible modules see [3, remarks and examples (3.2) (6)].
3. Z as Z -module is ($Z.S.R.$) module, because it's Z -small compressible module and by (1).
4. Every simple R -module is ($Z.S.R.$) module because it's Z -small compressible module [3, remarks and examples (3.2) (8)] and by (1).
5. The converse of (1) is not true because Z as a Z -module is a ($Z.S.R.$) module, but not simple.
6. Every retractable R -module is ($Z.S.R.$) R -module.
7. The converse of (6) does not hold for example, consider the following: we show that $A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$, $a, b \in R$ is not retractable, however the only Z -small submodule of A is the zero submodule, thus A is ($Z.S.R.$), [5].
8. Every compressible module is ($Z.S.R.$) module because every compressible module is retractable [5, remarks and examples (1.2.2) P6] and by (6).
9. The converse of (8) is not true for instance, Z_4 is ($Z.S.R.$) but not compressible module see [5, remarks and examples (1.1.2)(3)].
10. Every semi-simple R -module is ($Z.S.R.$) because it is retractable.
11. A homo-morphic image of a ($Z.S.R.$) module is a ($Z.S.R.$) module.
12. Let C be an R -module. Then C is a ($Z.S.R.$) R -module if and only if C is a ($Z.S.R.$) $R/ann C$ -module.

Proposition (2.3):

A Z -small submodule of ($Z.S.R.$) module is also ($Z.S.R.$) module.

Proof:

Let K a non-zero Z -small submodule of C and C be ($Z.S.R.$) module and let $0 \neq L \leq K \ll_Z C$, then $L \ll_Z C$ [1]. Since C is ($Z.S.R.$), so there exist a homo-morphism $f: C \rightarrow L$ and $i: K \rightarrow C$ is the inclusion homo-morphism, therefore $f \circ i: K \rightarrow L$ is a homo-morphism. Then K is a ($Z.S.R.$) module.

Proposition (2.4):

An R -module C is ($Z.S.R.$) if and only if for each $0 \neq x \in C$ with $\langle x \rangle \ll_Z C$, $Hom_R(C, Rx) \neq 0$.

Proof:

($\Rightarrow$) Suppose that C is ($Z.S.R.$) and let $0 \neq x \in C$. Put $N = \langle x \rangle = Rx$, then $0 \neq N \ll_Z C$ and $Hom_R(C, N) \neq 0$. Hence $Hom_R(C, Rx) \neq 0$.

($\Leftarrow$) To prove C is ($Z.S.R.$). Let $N \ll_Z C$ and let $0 \neq x \in N$, then $Rx \ll_Z N$, so by hypothesis, $Hom(C, Rx) \neq 0$, which implies that $Hom(C, N) \neq 0$ and therefore C is ($Z.S.R.$).

Lemma (2.5) [5]:

Every faithful cyclic R -module is retractable.

Proposition (2.6):

Every faithful cyclic R -module is ($Z.S.R.$).

Proof:

By lemma (2.5) and by Remarks and Examples (2.2,6) hence is ($Z.S.R.$).

Proposition (2.7) [1]:

Let $f: C \rightarrow \hat{C}$ be an homo-morphism. If $N \ll_Z C$ then $f(N) \ll_Z \hat{C}$.

Proposition (2.8):

Let C_1 and C_2 be isomorphic R -mod. Then C_1 is ($Z.S.R.$) if and only if C_2 is ($Z.S.R.$).

Proof:

Let $0 \neq N \ll_Z C_1$ and suppose that C_2 is (Z.S.R.). Let $f: C_1 \rightarrow C_2$ be an iso-morphism. Then by proposition (2.7), $0 \neq f(N) \ll_Z C_2$. Put $K = f(N) \ll_Z C_2$, we get $h: C_2 \rightarrow K$ is a homo-morphism (by assumption), let $g = f^{-1}|_K$, then $g: K \rightarrow C_1$ is a mono-morphism. $g(K) = f^{-1}(f(N)) = N$. Hence we have a composition $H = g \circ h \circ f$. Hence, $H: C_1 \rightarrow N$ is a mono-morphism. Then, C_1 is (Z.S.R.) module.

Proposition (2.9):

Let $C_1 \oplus C_2$ be an R -module such that $\text{ann}_R C_1 + \text{ann}_R C_2 = R$. Then $C_1 \oplus C_2$ is (Z.S.R.), if and only if C_1 and C_2 are (Z.S.R.) modules.

Proof:

($\Rightarrow$) Clear by proposition (2.8)

($\Leftarrow$) To prove $C_1 \oplus C_2$ is (Z.S.R.). Let $0 \neq N \ll_Z C_1 \oplus C_2$. As $\text{ann}_R C_1 + \text{ann}_R C_2 = R$ Thus by [6], $N = K_1 \oplus K_2$, for some $0 \neq K_1 \leq C_1$ and $0 \neq K_2 \leq C_2$. Since $N = K_1 \oplus K_2 \ll_Z C_1 \oplus C_2$ so by [1, proposition (2.7)] we have $K_1 \ll_Z C_1$ and $K_2 \ll_Z C_2$. Therefore $\text{Hom}(C_1, K_1) \neq 0$ and $\text{Hom}(C_2, K_2) \neq 0$. Let $0 \neq f: C_1 \rightarrow K_1$ and $g: C_2 \rightarrow K_2$. Now, define $h: C_1 \oplus C_2 \rightarrow K_1 \oplus K_2$ by $h(x, y) = (f(x), g(y))$, it is clear that h is a homo-morphism. If $h = 0$, then $h(x, y) = 0$ for all $x \in C_1, y \in C_2$, so $f(x) = 0$ and $g(y) = 0$ for all $x \in C_1, y \in C_2$, which is a contradiction since $f \neq 0$ and $g \neq 0$. hence $\text{Hom}(C_1 \oplus C_2, N) \neq 0$. Then $C_1 \oplus C_2$ is (Z.S.R.).

Recall that an R -mod C is Z -hollow if every proper submodule of C is Z -small [1].

In the following propositions (2.10) and (2.11), we give a sufficient condition for a (Z.S.R.) module to be retractable. So we get the converse of (6) in Remark and Example (2.2).

Proposition (2.10):

Let C be a (Z.S.R.) module. If every non-zero submodule of C contains a nonzero Z -small sub-module, then C is a retractable module.

Proof:

Let $0 \neq N \leq C$. By hypothesis, N contains a non-zero Z -small sub-module. Let $0 \neq K \ll_Z N$. Thus $K \ll_Z C$, by [1]. Hence $\text{Hom}(C, K) \neq 0$ (since C is (Z.S.R.) module), and therefore $\text{Hom}(C, N) \neq 0$. hence C is retractable module.

Proposition (2.11):

Let C be a Z -hollow module; then the following are equivalent:

- (1) C is a retractable module.
- (2) C is the (Z.S.R.) module.

Proof:

(1) $\Rightarrow$ (2) It's clear that see remarks and examples (2.2) (6).

(2) $\Rightarrow$ (1) Let $0 \neq N \leq C$ and C be Z -hollow module. Thus $N \ll_Z C$, but C is (Z.S.R.) module, then there exists a homo-morphism $f: C \rightarrow N$ and $\text{Hom}(C, N) \neq 0$. Therefore C is retractable module.

Proposition (2.12):

An R -module C is (Z.S.R.) if and only if there exists $0 \neq f \in \text{End}_R(C)$ such that $\text{Im } f \subseteq N$ for each non-zero Z -small submodule N of C .

Proof:

($\Rightarrow$) Suppose that C is (Z.S.R.). let $0 \neq N \ll_Z C$. Then $\text{Hom}_R(C, N) \neq 0$. Let $g: C \rightarrow N$ be a non-zero homo-morphism and $f = i \circ g$ where $i: N \rightarrow C$ be the inclusion homo-morphism, then $0 \neq f \in \text{End}_R(C)$ since $g \neq 0$ and i is a mono-morphism. Clearly, $f(N) = g(N) \subseteq N$.

($\Leftarrow$) Let $0 \neq N \ll_Z C$. By hypothesis, there exists a nonzero endo-morphism $f: C \rightarrow C$ and $f(C) \subseteq N$. Therefore, $f: C \rightarrow N$ is a nonzero homo-morphism this completes the proof.

3. Z-Small Retractable Modules and Z-Small Quasi-Dedekind Modules

In this section we will study the relationship between the concepts of Z -Small Quasi-Dedekind, Z -Small retractable (Z.S.R.), and some other concepts.

Definition (3.1)[4]:

An R -module C is called a Z -small quasi-Dedekind module if for all $f \in \text{End}(C)$, $f \neq 0$ implies $\text{Ker } f \ll_Z C$.

Remarks (3.2)[4]:

- (1) Every quasi-Dedekind is a Z -small quasi-Dedekind.
- (2) Every small quasi-Dedekind is Z -small quasi-Dedekind.

Proposition (3.3):

If C is Z -small quasi-Dedekind module, then C is not compressible.

Proof:

Consider C to be a Z -small quasi-Dedekind module. and let $N = \text{Ker } f \leq C$, but C is Z -small quasi-Dedekind, then $\text{Ker } f \ll_Z C$, $f \neq 0$, thus can't be embedded C in $\text{Ker } f$, because $\text{Hom}(C, \text{Ker } f) = 0$. Therefore C can't be compressible module.

Proposition (3.4):

If C is Z -small quasi-Dedekind R -module, then C is not $(Z.S.R.)$.

Proof:

Suppose that C is Z -small quasi-Dedekind module and let $N = \text{Ker } f \leq C$, but C is Z -small quasi-Dedekind, then $\text{Ker } f \ll_Z C$, $f \neq 0$, thus $\text{Hom}_R(C, \text{Ker } f) = 0$. Therefore C is not $(Z.S.R.)$ module.

As it noted in the example and remark (3.2, 1) that any Z -small compressible module is $(Z.S.R.)$ and the converse need not be true in general, we recall in the following result that the opposite applies under specific conditions:

Proposition (3.5):

If C is a $(Z.S.R.)$ quasi-Dedekind R -module, then every nonzero element of $\text{Hom}(C, N)$ is a mono-morphism for any nonzero Z -small submodule N of C .

Proof:

Let $0 \neq N \ll_Z C$ and let $f: C \rightarrow N$ be a nonzero homo-morphism. Hence $\text{iof} \in \text{End}(C)$ and $\text{iof} \neq 0$. For if $\text{iof} = 0$, then $\text{if}(C) = f(C) = 0$ implies $f = 0$ which is a contradiction. Then $0 \neq \text{iof} \in \text{End}(C)$ and by hypothesis iof is a mono-morphism which gives that f is a mono-morphism.

Corollary (3.6):

Let C be a $(Z.S.R.)$ module. If C is quasi-Dedekind, then C is Z -small compressible.

Proposition (3.7)[3]:

Let $C = Rm_1 \oplus Rm_2 \oplus \dots \oplus Rm_k$, where $m_1, m_2, \dots, m_k \in C$. If C is Z -small uniform and Z -small prime, then C is a Z -small compressible module.

Proposition (3.8)[3]:

If C is Z -small compressible R -module, then C is Z -small prime and C is Z -small uniform.

The following result connects $(Z.S.R.)$, Z -small prime and Z -small uniform under the condition finitely generated quasi-Dedekind:

Corollary (3.9):

Let C be a finitely generated quasi-Dedekind R -module; then C is $(Z.S.R.)$ if and only if C is Z -small prime and Z -small uniform.

Proof:

Then C is Z -small compressible based on corollary (3.6), and the result is obtained using propositions (3.7) and (3.8).

Recall that an R -module C is called Z -small monoform module, if every nonzero partial endo-morphism of C has Z -small kernel[7].

Remark (3.10)[8]:

Every Z -small compressible R -module is a Z -small monoform, but not conversely. For example, Z_8 as a Z -module is Z -small monoform but not Z -small compressible.

Proposition (3.11):

Let C be a quasi-Dedekind R -module. Then C is Z -small monoform if and only if C is Z -small compressible.

Proof:

($\Rightarrow$) Suppose that C is Z -small monoform. Let $0 \neq N \ll_Z C$, then $0 \neq f \in \text{Hom}_R(N, C)$. Since C is quasi-Dedekind, then $f \circ g: C \rightarrow N \rightarrow C$ is a mono-morphism, hence $g: C \rightarrow N$ is a mono-morphism. Thus C is Z -small compressible.

($\Leftarrow$) It is clear by remark (3.10).

Corollary (3.12):

Let C be a $(Z.S.R.)$ quasi-Dedekind module; then C is a Z -small monoform.

4. Conclusion

In this paper, we successfully introduced the concept of Z -small retractable, and we show that it is an expansion of the concept of small retractable modules. We Also presented some important relations between Z -small retractable modules and other kinds of modules.

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