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Common Fixed Points and Invariant Best Approximations in 2-Banach Spaces

Rafah Sajid Abed Ali
Office of University President, University of Information Technology and Communications, Baghdad, Iraq.

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ABSTRACT

In this paper, we introduce and study the concepts of 2-f-contraction and 2-f-nonexpansive mappings in the framework of 2-normed spaces. The main objective is to generalize classical contraction-type mappings and provide an effective approach for analyzing nonlinear problems in such spaces. The methodology is based on extending classical fixed point techniques and employing the properties of 2-normed and 2-Banach spaces under suitable contraction conditions. We establish the existence and uniqueness of a common fixed point for two commuting mappings. In addition, two main results concerning invariant best approximation in 2-Banach spaces are obtained. The results extend and improve several known theorems and demonstrate the applicability of the proposed approach in fixed point theory and approximation problems in 2-normed and 2-Banach spaces.

Corresponding Author:

Rafah Sajid Abed Ali

Office of University President,
University of Information Technology and Communications, Baghdad City, Iraq.
Email: *rafahsa87@gmail.com*

1. INTRODUCTION

The concept of a common fixed point arises when two or more mappings share a point that remains invariant under each mapping. The existence and uniqueness of such points play an important role in solving systems of operator equations and related nonlinear problems. In contrast, best approximation theory deals with the problem of finding an element in a subset that is nearest to a given point, which is a fundamental topic in approximation theory. When such a best approximant remains invariant under a given mapping, it is referred to as an invariant best approximation. Recently, there has been significant interest in studying the relationship between common fixed points and invariant best approximations in normed and Banach spaces [1], [2], [6]. These investigations extend classical results and contribute to the development of approximation theory in more abstract frameworks. Moreover, the interaction between fixed point theory and approximation theory provides powerful tools for analyzing nonlinear mappings and understanding the geometric structure of 2-normed spaces.

Brosowski [3], Meinardus [6], and Singh [10] obtained several important results on invariant approximation in normed spaces using fixed point techniques. Jungck and Sessa [4] also developed related results in normed space settings. Their work has been extended and generalized by many authors; see, for example, Habiniak [6], Hicks and Humphries [12], Latif and Bano [1], Narang [11], and Smoluk [2]. Singh [10], in particular, presented a unified approach in locally convex spaces, which has motivated further research in generalized settings.

The present paper investigates common fixed points and invariant best approximations in 2-normed spaces and 2-Banach spaces [5], [7], and establishes new existence and uniqueness results under appropriate conditions. Recently, several studies have extended fixed point theory and invariant approximation to various generalized settings [13–17].

2. Preliminaries

In this section, we recall some basic concepts that will be used later.

Definition 2-1: [7]

Let X be a real linear space and $\|.,.\|$ be a non-negative real-valued function defined on $X \times X$ satisfying the following conditions:

1) $\|x, y\| = 0$ if and only if x and y are linearly dependent in X.

2) $\|x, y\| = \|y, x\|$, for all x ,y $\in X$.

3) $\|x, \alpha y\| = |\alpha|\|x, y\|$, α ϵ R ,$x$, y ϵ X.

4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for all $x , y , z \in X$.

Then $(X ,\|.,.\|)$ is called a **2-normed space**.

Note that the 2-normed space is Hausdorff space and $\|.,.\|$ is continuous function, for examples of 2-normed spaces, see [7] .

Now, we give the definition of the coincidence point:

Definition 2-2:

Let X be a 2-normed space, an element $x \in$ X is called a **coincidence point of f and T** if $fx = Tx$, we denote $C(f \cap T)$ the set of coincidence points of f and T.

Definition 2-3: [7]

A sequence $\{x_n\}$ in a 2-normed space (X,‖.,.‖) is called a **convergent sequence** if there is, x ϵ X, such that $\lim_{n\to\infty}\|x_n - x, u\| = 0$, for all $u \in X$.

Definition 2-4: [7]

A sequence $\{x_n\}$ in a 2-normed space (X,‖.,.‖) is called a **Cauchy sequence** if $\lim_{m,n\to\infty} \|x_m - x_n, y\| = 0$, for all $y \in$ X.

Definition 2-5: [5]

A linear 2-normed space X is said to be complete if every Cauchy sequence is convergent to an element of X. Then X is called a **2-Banach space**.

Definition 2-6:

Let X be a 2-normed space, C ⊆ X and T : C → C be a mapping then T is called:

i. Continuous if for all $\{x_n\} \subset$ C such that $x_n \to x$ implies that $\{Tx_n\} \to Tx$.

ii. Weakly continuous if for all $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ implies that $\{Tx_n\} \rightharpoonup Tx$.

Definition 2-7: [9]

Let X be a 2-normed space, Then we say that X satisfies **Opial condition** if for every bounded sequence (x_n) ϵ X converges weakly to x ϵ X, Then $\lim_{n\to\infty} inf \|x_n - x, u\| < \lim_{n\to\infty} inf \|x_n - y, u\|$ for every $x \neq y$ & $y, u \in X$.

Definition 2-8: [9]

Let X be a 2-normed space and C ⊆ X, then a mapping T : C → X is said to be **2-demi closed** at $y \in$ X if for any sequence $\{x_n\}$ in C, such that $x_n \rightharpoonup x \in$ C and T$x_n \to y$ Imply, T$x = y$.

Definition 2-9: [10]

Let X be a 2-normed space, T : X ⟶ X be a mapping, Then T is said to 2-**f-contraction mapping** if for a fixed constant k; $k \in [0,1]$ and for all $x, y \in$ X

$\|Tx - Ty, u\| \leq k\|fx - fy, u\|, u \in$ X………. (2.1)

If $k = 1$, then T is called **2-f-non-expansive mapping** s.t.,

$\|Tx - Ty, u\| \leq \|fx - fy, u\|$………. (2.2)

Definition 2-10: [11]

Let X be a 2-normed space, **the weak topology** on X is the topology generated by the sets V(x; f; ε)=$\{y \in$ X : $|f(x - y), u| <$ ε}, where $x, u \in$ X ,$f \in$ X* and ε > 0.

Definition 2-11: [11]

Let X be a 2-normed space and let C be a subset of X, then C is said to be **weakly compact** if C is compact in the weak topology.

Definition 2-12: [4]

Let X be a 2-normed space, C ⊆ X, Then C is called **2-q-starshaped** with $q \in$ C if $kx + (1 - k)q \in$ C, For all $x \in$ C and all $k \in [0,1]$.

Definition 2-13: [4]

Let X be a 2-normed space, C ⊆ X, Then C is called **affine** if C is convex and $f(kx + (1 - k)y) = kfx + (1 - k)f$y, for all $x, y \in$ C and all $k \in [0,1]$.

Definition 2-14:

Let X be a 2-normed space, C ⊆ X and $x, u \in X$. An element $y \in$ C is called an **element of best approximation to x** if

$$\|x - y, u\| = inf_{z\in C} \|x - z, u\| \text{ ……….(2.3)}$$

The set of best approximation to x out of C is denoted by $P_C(x)$, i.e.: $P_C(x) = \{y \in C: \|x - y, u\| = inf_{z \in C}\|x - z, u\|\}$.

Table1: Summary of Definitions and Concepts

No.	Concept	Description
2.1	2-Normed Space	A real linear space X with a function ‖.,.‖ satisfying standard properties.
2.2	Coincidence Point	A point x such that $F(x) = T(x)$.
2.3	Convergent Sequence	‖x_n- x, u‖ → 0 for all $u \in X$.
2.4	Cauchy Sequence	‖x_m -x_n, y‖ → 0 as $m, n \to \infty$.
2.5	2-Banach Space	A complete 2-normed space.
2.6	Continuous Mapping	x_n→ x implies $T(x_n)$ →$T(x)$.
2.6(ii)	Weakly Continuous	x_n ⇀ x implies $T(x_n) \rightharpoonup T(x)$.
2.7	Opial Condition	Condition ensuring uniqueness of weak limits.
2.8	2-Demi Closed Mapping	x_n ⇀ x and $T(x_n) \to y$ implies $T(x) = y$.
2.9	2-f-Contraction	$\|Tx - Ty\| \leq k\|fx - fy\|$.
2.10	Weak Topology	Generated by functionals in X^*.
2.11	Weakly Compact Set	Compact in weak topology.
2.12	Starshaped Set	$kx - (1 - k)q \in C$.
2.13	Affine Set	Preserves convex combinations.
2.14	Best Approximation	Minimizes distance over C.

3. RESULTS AND DISCUSSION

In this section, we present the main results concerning common fixed points and invariant best approximation in the 2-Banach spaces. The introduced concepts of 2-f-contraction and 2-f-nonexpansive mappings, studied in the literature [5], [7], [10], are used to establish several new results that extend and improve existing theorems in the literature [3], [6], [8], [10]. In particular, we investigate the existence and uniqueness of common fixed points for commuting mappings and study the invariant approximation properties under suitable conditions.

<u>Theorem (3-1)</u>

Let X be a 2-Banach space. Let T : X → X and f: X → X be a continuous mapping which commutes with T and $T(X) \subseteq f(X)$, suppose that there exists $k \in (0,1)$, $u \in$ X Such that $\|Tx - Ty, u\| \leq k\|fx - fy, u\|$, for each $x, y \in$ X.Then there existence of a unique common fixed point for T and f.

Proof.

For x_o, construct the sequences x_n , y_n in X as T$x_n = fx_n = y_n$ for all $n \in$ X.Firstly, we prove { y_n} is Cauchy sequence in X.

$$\begin{aligned} \| y_n - y_{n+1}, u\| &= \|Tx_{n-1} - Tx_n, u\| \\ &\leq k\|fx_{n-1} - fx_n, u\| \\ &= k\|Tx_{n-2} - Tx_{n-1}, u\| \\ &\leq k^2\|fx_{n-2} - fx_{n-1}, u\| \\ &\leq \dots \\ &\leq k^{n-1}\|fx_o - fx_1, u\| \\ &= k^{n-1}\|y_o - y_1, u\| \end{aligned}$$

Now, when $n > m$ we have

$$\begin{aligned} \|y_m - y_n, u\| &\leq \|y_m - y_{m-1}, u\| + \cdots + \|y_{n+1} - y_n, u\| \\ &\leq k^{n+1}(k^{m-n-2} + \cdots + 1)\|y_o - y_1, u\| \\ &= \frac{k^{n+1}}{1-k}\|y_o - y_1, u\|, \end{aligned}$$

And this show that $\{y_n\}$ is Cauchy sequence. since X is complete then $y_n = fx_n = Ty_{n-1} \longrightarrow w \in$ X.
By the continuity of f and f ,T are commute, we get $f^2x_n \longrightarrow fw$ And $Tfx_n \longrightarrow Tw$
Putting $x = fx_n$ and $y = Tx_n$ in equation (2.1) we have

$$\begin{aligned} \|Tx - Ty, u\| &= \|Tfx_n - TTx_n, u\| \\ &\leq k\|fx_n - Tx_n, u\| \end{aligned}$$

When $n \longrightarrow \infty$, $\|Tw - w, u\| \leq k\|w - Tw, u\|$
So, T$w = w$
By similar way show that $fw = w$.
For uniqueness, let z be another common fixed point, i.e., $z = Tz = fz$ then

$$\|Tz - Tw, u\| \leq k\|fz - fw, u\|$$

And $\|Tx_n - Tw, u\| \leq k\|fx_n - fw, u\|$
Therefore,

$$\|z - w, u\| \leq k\|z - w, u\|$$

As $n \to \infty$, which mean $z = w$.

This theorem extends classical fixed point results to the setting of 2-Banach spaces and ensures the uniqueness of the common fixed point under a generalized contraction condition.

Lemma (3-2):

Let X be a 2-Banach space, C be weakly compact subset of X satisfying Opial's condition. Let f : C → X be weakly continuous map and T : X → X satisfies equation (2.2). Then $f - T$ is 2-demi closed.

Proof.

Let $\{x_n\}$ be a sequence in C and

$y_n = fx_n - Tx_n$ ………. (3.1)

$x_n \rightharpoonup x \in$ C and $y_n \longrightarrow y$ and $fx_n \rightharpoonup fx$.

By using Equation (2.2) of T, we have

$$\|Tx_n - Tx, u\| \le \|fx_n - fx, u\|; u \in \text{C} \quad \text{……….} \ (3.2)$$

From (3.1) and (3.2), by taking the limit with respect to n , we obtain

$$\lim_{n\to\infty} inf\|Tx_n - Tx, u\| \le \lim_{n\to\infty} inf\|fx_n - fx, u\|$$

$$\lim_{n\to\infty} inf\|fx_n - y_n - Tx, u\| \le \lim_{n\to\infty} inf\|fx_n - fx, u\| \text{……….} \ (3.3)$$

Tx is compact and $y_n \to y$,

$$\lim_{n\to\infty} inf\ \|fx_n - y - Tx, u\| \le \lim_{n\to\infty} inf\ \|fx_n - fx, u\|.$$

Since X satisfies Opial's condition and $fx_n \rightharpoonup fx$, we obtain $fx = y + Tx$.

Thus,

$$y = fx - Tx = (f - T)\, x$$

Which proves that $f - T$ is 2-demi closed.

This lemma plays a crucial role in establishing the demi-closedness property, which is essential in proving the existence of coincidence points in the subsequent results.

Theorem (3-3):

Let X be a 2-Banach space and C be a weakly compact subset of X which is 2-q-starshaped with respect to $q \in$ C. Let f : C → C be a weakly continuous and affine map such that fC = C and $fq = q$ and let T : C → C satisfies Equation (2.2) which commutes with f. If $(f -$ T) is 2-demi closed on X or X satisfies Opial's condition then C($f \cap$ T) $\neq \emptyset$.

Proof.

Let $\{k_n\}$ be a sequence of real numbers such that $0 < k_n < 1$ and $k_n \to 1$ as $n \to \infty$. Define a mapping T_n by setting $T_n x = k_n Tx + (1 - k_n)\, q$

For all $x \in C$. Since T: C → C, we observe that for each $n \ge 1$, T_n : C → C, since C is 2-q-starshaped and T(C) ⊆ C. Also T_n(C) ⊆f(C). Let $x, y \in$ C.

Then

$$\|T_n x - T_n y, u\| = k_n\|Tx - Ty, u\|, u \in \text{X}.$$

By Equation (2.2) of T we have

$$\|T_n x - T_n y, u\| \le k_n\|fx - fy, u\|$$

Showing that T_n satisfies equation (2.1).

Now, we show that T_n commutes with f.

$$\begin{aligned} T_n(fx) &= k_n Tfx + (1 - k_n)fq \\ &= k_n fTx + (1 - k_n)fq \\ &= f(k_n Tx + (1 - k_n)q) \\ &= f(T_n x). \end{aligned}$$

For each $x \in$ C. The above holds since $fq = qf$, $fT = Tf$ and f is an affine. Thus $T_n f = fT_n$ for each n ≥ 1. Thus all the conditions of Theorem (3-1) are satisfied and hence there is an $x_n \in$ C,Such that f $x_n = T_n x_n$, So by the definition of $T_n x_n$,

$$\begin{aligned} fx_n &= k_n Tx_n + (1 - k_n)q \\ fx_n - Tx_n &= k_n\, Tx_n + (1 - k_n)q - Tx_n \\ &= (1 - k_n)(q - Tx_n) \\ &= \left(\frac{1}{k_n} - 1\right)(q - fx_n). \end{aligned}$$

And

$$\|fx_n - Tx_n, u\| \le (\tfrac{1}{k_n} - 1)\{\|g, u\| + \|fx_n, u\|\}.$$

Since T(C) ⊆ C is bounded and $fx_n = T_n x_n \subseteq$ C, we have $\|fx_n, u\|$ is bounded so by the fact that $k_n \to 1$, we have $\|fx_n - Tx_n, u\| \to 0$,Since C is weakly compact, there is a subsequence $\{x_{n_i}\}$ of sequence which $x_{n_i} \rightharpoonup x_o \in$ C,

As $fx_{n_i} - Tx_{n_i} \in (f - T)x_{n_i}$

If $f - T$ is 2-demi closed, then
$0 \in (f - T)\, x_o$ and hence $f x_o = T x_o$, If X satisfies Opial's condition, then it follows from Lemma (3-2) that $f - T$ is 2-demi closed and hence f and T have a coincidence point $x_o \in$ C as in the previous case. The above theorem guarantees the existence of a coincidence point under weaker assumptions, which generalizes several known results in fixed point theory.

<u>Theorem (3-4):</u>

Let X be a 2-Banach space. Let f: X → X be a weakly continuous map and let T: X →X satisfies Equation (2.2) such that $Tx_o = \{x_o\}$ for some $x_o \in$ X Let C be a nonempty T-invariant subset of X and $x_o \in$ F(T) ∩ $F(f)$ assume that $P_C(x_o)$ is nonempty, weakly compact and 2-q-starshaped with respect to $q \in$ F(f). Further assume that f is continuous and affine map on $P_C(x_o)$ with $f\big(P_C(x_o)\big) = P_C(x_o)$, If f −T is 2-demi closed on $P_C(x_o)$, Then $P_C(x_o)$ ∩F(T) ∩ $F(f) \neq \emptyset$.

Proof.

Let D = $P_C(x_o)$ and let $a \in$ D, then $a \in$ C and

$$\|x_o - a, u\| = \|x_o - C, u\|;\ u \in \text{X}.$$

Let $b \in Ta \subset$ C. then we have

$$\|b - x_o, u\| \leq \|Ta - Tx_o, u\|$$

So, by using the Equation (2.2) of T, we get

$$\|Ta - Tx_o, u\| \leq \|fa - fx_o, u\|$$
$$\|b - x_o, u\| \leq \|fa - fx_o, u\|$$

As $fx_o = x_o$ and $fa \in P_C(x_o)$, we have

$$\|b - x_o, u\| \leq \|fa - x_o, u\| = \|x_o - C, u\|$$

Which gives that $b \in$ D and thus $Ta \subset$ D. therefore T: D → D. Now let q be the star center of D. then for each $x \in$ D and any $k \in (0,1), kx + (1-k)q \in$ D.
Take $\{k_n\}$ sequence of real numbers such that $k_n \in (0,1)$ and $k_n \to 1$ as $n \to \infty$. Now for each n define a map T_n by setting $T_n x = k_n Tx + (1-k)q$, for all $x \in$ D.
Clearly, each T_n : D ⟶ D, Since f is continuous, affine and commute with T, we have

$$\begin{aligned} T_n\, fx &= k_n Tfx + (1-k)fq \\ &= k_n fTx + (1-k)fq \\ &= f(k_n Tx + (1-k)q) \\ &= fT_n\, x. \end{aligned}$$

Thus each T_n commutes with f for each n and $T_n(D) \subseteq D = f(D)$. Let $x, y \in D$. Then by the definition of T_n and the Equation (2.2) of T, we have

$$\|T_n x - T_n y, u\| = k_n \|Tx - Ty, u\| \leq k_n \|fx - fy, u\|,$$

Which proves that each T_n satisfies Equation (2.1). Also, since D is a 2-Banach space, it follows from Theorem (3.1) that for each $n \geq 1$, there exists $x_n \in D$
Such that $fx_n = T_n x_n$, such that

$$\begin{aligned} fx_n &= k_n y_n + (1-k_n)q \\ fx_n - Tx_n &= (1-k_n)q + (1-k_n)Tx_n \\ &= (1-k_n)(q - Tx_n) \\ &= \left(\frac{1}{k_n} - 1\right)(q - fx_n) \end{aligned}$$

And $\|fx_n - Tx_n, u\| = \left(\frac{1}{k_n} - 1\right)\|q - fx_n, u\| \longrightarrow 0$ as $n \to \infty$.

Since D is weakly compact, there exists a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, and we have $x_n \rightharpoonup z \in D$. Now as $fx_n - Tx_n \in (f - T)\, x_n$ and $f - T$ is 2-demi closed, we conclude that $0 \in (f - T)z$, And hence $f(z) = T(z)$.

This result shows the existence of invariant best approximation under the given conditions and highlights the applicability of the proposed method in approximation theory.

<u>Corollary (3-5)</u>:

Let X be a 2-Banach space satisfying Opial's condition Let f: X → X be a weakly continuous map and T : X → X satisfies Equation (2.2) such that $Tx_o = x_o$ for some $x_o \in$ X, Let C be a nonempty T-invariant subset of X and $x_o \in$ F(T) ∩ F(f), Assume that $P_C(x_o)$ is non-empty, weakly compact and 2-q-starshaped with respect to q ∈ F(f). Further, if f is continuous and an affine map on $P_C(x_o)$ with $f(P_C(x_o)) = P_C(x_o)$, then $P_C(x_o) \cap F(f) \cap F(T) \neq \emptyset$.

Proof.

Let D = $P_C(x_o)$ and let $a \in$ D. then $a \in$ C and $\|x_o - a, u\| = \|x_o - C, u\|, u \in$ C
So, by using the Equation (2.2) of T, we get

$$\|Ta - Tx_o, u\| \leq \|fa - fx_o, u\|.$$

Thus,

$$\|Ta - x_o, u\| \leq \|fa - fx_o, u\| = \|x_o - C, u\|,$$

Therefore, T : D ⟶ D, Thus, by Lemma (3-2), we have f − T is 2-demi closed on D. Now the result follows from Theorem (3-4).

Table 2: Main Results in 2-Banach Spaces

No.	Result	Description
3.1	Fixed Point Theorem	Existence and uniqueness of common fixed point.
3.2	Lemma (Demi-Closedness)	f - T is 2-demi closed.
3.3	Coincidence Point Theorem	$C(f \cap T) \neq \emptyset$.
3.4	Approximation Theorem	Existence $P_C(x_o)$
3.5	Corollary	Extension under Opial condition.

4. CONCLUSION

In this paper, we investigated common fixed point problems and invariant best approximation in the framework of 2-Banach spaces. We introduced the concepts of 2-f-contraction and 2-f-nonexpansive mappings and studied their properties under suitable conditions. We established the existence and uniqueness of a common fixed point for two commuting mappings. In addition, we proved several results related to invariant best approximation and showed the 2-demi closedness of the operator $f - T$, which plays an important role in deriving coincidence point results. The obtained results extend and generalize several known results in the literature and provide a useful framework for studying nonlinear mappings in 2-normed and 2-Banach spaces. These findings contribute to the development of fixed point theory and approximation theory in such spaces.

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BIOGRAPHIES OF AUTHORS

Rafah S. Abed Ali is a lecturer at the University of Information Technology and Communications, Iraq. She holds a M.Sc. degree in Mathematics from the College of Education Ibn Al-Haitham, University of Baghdad, Iraq, with specialization in Functional Analysis. Her research interests include fixed point theory, invariant approximation, 2-normed spaces, 2-Banach spaces, and nonlinear functional analysis. She has published one scientific paper in the field of mathematical analysis. She can be contacted at email: rafahsa87@gmail.com.