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Efficient Fuzzy Solutions for Linear Fractional Programming Problems: Using Pentagonal Fuzzy Numbers and the Decomposition Method

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ABSTRACT

This paper presents an efficient method for solving fully fuzzy linear fractional programming (FFLFP) problems using pentagonal fuzzy numbers (PFNs) and the decomposition method. In real-world decision-making, uncertainty is inevitable, and fuzzy set theory provides a powerful framework to model such uncertainty. The proposed approach decomposes the FFLFP problem into five independent linear fractional programming (LFP) problems. Each LFP is transformed into a linear programming (LP) problem using the Charnes and Cooper transformation and solved via the simplex method. The obtained solutions are then aggregated to construct an efficient fuzzy solution. A numerical example is provided to demonstrate the effectiveness and applicability of the method. The results show that the proposed technique is accurate and suitable for solving complex optimization problems under uncertainty.

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1. INTRODUCTION

The implementation process in the real world of decision-making problems is often uncertain. In general, in the present time, due to the hustle and bustle of society, scientists and researchers in the field of optimization problems often resort to uncertainty sets, especially fuzzy sets, to deal with these problems. Due to market conditions and rapid changes in measurements, or because some problems cannot be controlled, PFNs are required in this case. This paper uses this paper for all the coefficients and decision variables of the fully fuzzy linear fractional programming problems.

Such problems have rarely been addressed, so we thought it necessary to propose a way to solve them scientifically and practically. This method can be used in many modern business companies, such as oil companies, soft drink companies, pharmaceutical companies, etc.

With the proposed method, all possibilities regarding purchases, sales, costs, profits, and wages can be considered and given importance. In the past decades, several novel methods and techniques have emerged to solve LFP problems, such as [1], which is used as a basis in this paper for objective formats [2] and [3], each of which is in different forms and for different coefficients. Then, for the case of fuzzy linear fractional programming problems [4, 5, 6, 7] worked on the implementation and solution of life problems. However, it is very workable when the coefficients are rough intervals [8]. For the situation where the coefficients of linear programming (LP) problems are PFNs [9], both used two new algorithms to solve it. Authors [10, 11] and others have also recently worked on fractional programming problems with PFN coefficients.

In the present study, we have suggested an interesting solution to a rare type of PFLFP problem when we use the decomposition method as a basis, with some restrictions that, according to several definitions and theorems. In the situation of multi-objective linear fractional programming problems, Charnes & Cooper's method is used for each objective function separately, which is a successful and accurate way to identify the optimal solution. The order of the created objectives then determines the effective fuzzy solution.

Then, using the simplex method, a fuzzy optimal solution was found, and the proposed technique was compared with the technique of Sahoo et al.[10]. The results are shown in Table 1. The paper is organized as follows: Section 2 deals with fuzzy numbers, PFNs, and their mathematical processes, also deals with mathematical formulations of the LFP problem and fully fuzzy linear fractional programming problems, with the translation procedure of PFLFP problems to five LFP problems using the decomposition method, discusses our technique, and supports and strengthens it by proving a theorem. Section 3 provides an appropriate numerical case to demonstrate our established method and shows the table comparison, also deals with the results and discussion. Section 4 deals with the conclusion.

2. METHODOLOGY

2.1 Preliminaries

Definition 2.1.1 [12]

A **universal set**, denoted by the symbol U, is a set that contains all possible elements or objects under consideration in a particular context.

Definition 2.1.2 [13]

Let U be a universal set, then a fuzzy subset $\tilde{A}$ of U where $\tilde{A}$ is the ordered pairs$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in U\}$, and $\mu_{\tilde{A}}: U \to [0,1]$ is known as the membership function $\tilde{A}$ is called a **fuzzy set**.

Definition 2.1.3 [14]

The support of a fuzzy set $\tilde{A}$ is the set of all points $x \in U$ such that $\mu_{\tilde{A}}(x) > 0$.

Definition 2.1.4 [15]

A fuzzy set $\tilde{A}$ is said to be **normal** if $\mu_{\tilde{A}}(x) = 1$ for at least one $x \in U$.

Definition 2.1.5 [15]

A fuzzy set $\tilde{A}$ on X is **convex** if and only if $\mu_{\tilde{A}}(\alpha x_1 + (1-\alpha)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), \forall\, x_1, x_2 \in U$and $\alpha \in [0,1]$.

Definition 2.1.6 [13]

$\tilde{A}$ is called a **fuzzy number**; if $\tilde{A}$ Is convex, normal, and supports of $\tilde{A}$ must be bounded.

Definition 2.1.7 [16]

The α -cut or α -the level set of a fuzzy set is a **crisp set** defined by $\tilde{A}_\alpha = \{x \in R /\ \mu_{\tilde{A}}(x) \geq \alpha\}$, where $\alpha \in\ [0,1]$.

Definition 2.1.8 [10]

A fuzzy set $\tilde{A}$ is called **PFNs** if its demonstration is in the form $\tilde{A} = (A_1,\ A_2, A_3, A_4, A_5)$ with $A_1 \leq A_2 \leq A_3 \leq A_4 \leq A_5$, and the membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - A_1}{A_2 - A_1}, & A_1 \leq x \leq A_2 \\ \dfrac{x - A_2}{A_3 - A_2}, & A_2 \leq x \leq A_3 \\ 1 & x = A_3 \\ \dfrac{A_4 - x}{A_4 - A_3}, & A_3 \leq x \leq A_4 \\ \dfrac{A_5 - x}{A_5 - A_4}, & A_5 \leq x \leq A_4 \\ 0, & otherwise \end{cases}$$

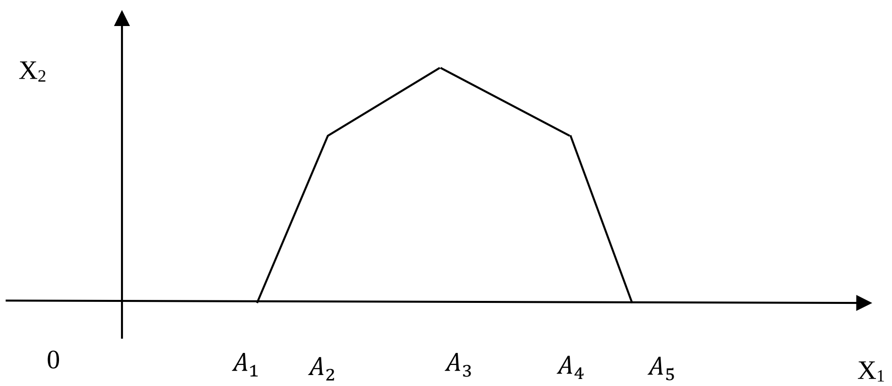

Figure 1. PFNs Membership Function.

Definition 2.1.9 [17]

Let $\tilde{A} = (A_1, A_2, A_3, A_4, A_5)$ and $\tilde{B} = (B_1, B_2, B_3, B_4, B_5)$ be two PFNs, where $A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, B_4, B_5 \in R$.

Then, the arithmetic operations and scalar multiplications are defined by

1- $\tilde{A} \oplus \tilde{B} = (A_1 + B_1, A_2 + B_2, A_3 + B_3, A_4 + B_4, A_5 + B_5)$

2- $\tilde{A} \ominus \tilde{B} = (A_1 - B_5, A_2 - B_4, A_3 - B_3, A_4 - B_2, A_5 - B_1)$

3- $L \otimes \tilde{A} = (LA_1, LA_2, LA_3, LA_4, LA_5)$

where $L \in R^{+}$ and $L \otimes \tilde{A} = (LA_5, LA_4, LA_3, LA_2, LA_1)$ where $L \in R^{-}$

4- $\tilde{A} \otimes \tilde{B} = \tilde{A}\tilde{B} = (A_1B_1, A_2B_2, A_3B_3, A_4B_4, A_5B_5)$

5- $\tilde{A} \oslash \tilde{B} = (\frac{A_1}{B_5}, \frac{A_2}{B_4}, \frac{A_3}{B_3}, \frac{A_4}{B_2}, \frac{A_5}{B_1})$, if one of the components of $\tilde{B}$ becomes zero, we cannot find its division.

Definition 2.1.10 [18]

A PFN $\tilde{A} = (A_1, A_2, A_3, A_4, A_5)$ is called **positive (negative)** PFN iff $A_1, A_2, A_3, A_4, A_5 \geq 0$ $(A_1, A_2, A_3, A_4, A_5 \leq 0)$

Definition 2.1.11 [19]

If x^0 is feasible, and there are no alternative solutions x^*, such as $Cx^0 \neq Cx^*$ and $Cx^0 \leq Cx^*$ for maximizing problems and $Cx^0 \geq Cx^*$ for minimizing problems, then x^0 is considered an efficient solution to a linear programming problem.

2.2 Mathematical Formulations

This section demonstrates how the suggested method uses the mathematical forms related to this study as its foundation.

2.2.1 Linear Fractional Programming (LFP) Problem:[20]

The LFP problem is usually expressed as follows:

$Max.z = \frac{aX+\alpha}{dX+\beta}$

subject to:

$AX^T \leq b, X \geq 0.$

where $a, d, X \in R^m, b \in R^n, A$ is an $n \times m$ real matrix and $\alpha, \beta \in R$.

2.2.2 Pentagonal Fuzzy Linear Fractional Programming (PFLFP) problems:[9, 21]

We will model the LFP problem using PFNs for all variables and coefficients.
The PFLFP problem has the following general form:

$Max.z = \frac{\tilde{a}\tilde{X}+\tilde{\alpha}}{\tilde{d}\tilde{X}+\tilde{\beta}}$

subject to:

$\tilde{A}\tilde{X} \leq \tilde{b}, \tilde{X} \geq 0.$

$\tilde{A} = (A_1, A_2, A_3, A_4, A_5), \tilde{a} = (a_1, a_2, a_3, a_4, a_5), \tilde{d} = (d_1, d_2, d_3, d_4, d_5),$
$\tilde{X} = (x, y, z, w, t), \tilde{b} = (b_1, b_2, b_3, b_4, b_5)$, and $\tilde{\alpha}, \tilde{\beta}$ ∈PFNs.

2.3 The Proposed Approach Derivation

This section discusses the suggested method, focusing on specific ideas. We present a derivation for solving a general form of the PFLFP problem and have proved the theorem that optimal solutions become efficient fuzzy solutions to the original problem.
Let us consider the PFLFP problem. The problem 1 below can be written as

Problem 1: $Max.z = Max.(V_1, V_2, V_3, V_4, V_5) = \frac{F(\tilde{X})}{G(\tilde{X})} = \frac{\sum_{j=1}^{m} \tilde{a}_j \tilde{X}_j \oplus \tilde{a}_{m+1}}{\sum_{j=1}^{m} \tilde{d}_j \tilde{X}_j \oplus \tilde{d}_{m+1}}$

subject to: (S)

$\sum \tilde{A}_{ij}\tilde{X}_j \leq \tilde{b}_i, \tilde{X}_j \geq 0.$ For $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

To avoid $G(\tilde{X}) = 0$, we will assume that $G(\tilde{X}) > 0$.

First, by taking advantage of the arithmetic operations of PFNs, problem 1 was decomposed and then converted into five independent LFP problems, as follows:

First Value $= Max.V_1 = \frac{\sum_{j=1}^{m} a_{1j}x_j}{\sum_{j=1}^{m} d_{5j}x_j}.$

Subject to: (E_1)

$$\sum_{j=1}^{m} A_{kj}x_j \ (\leq = \geq) b_{ki}$$

$x_j \geq 0$, for$k = 1, \ldots, 5$ and $i = 1, \ldots, n$.

Second Value $= Max.V_2 = \frac{\sum_{j=1}^{m} a_{2j}y_j}{\sum_{j=1}^{m} d_{4j}y_j}.$

Subject to: (E_2)

$$\sum_{j=1}^{m} A_{kj}y_j \ (\leq = \geq) b_{ki}$$

$y_j \geq 0$, for$k = 1, \ldots, 5$ and $i = 1, \ldots, n$.

Third Value $= Max.V_3 = \frac{\sum_{j=1}^{m} a_{3j}z_j}{\sum_{j=1}^{m} d_{3j}z_j}.$

Subject to: (E_3)

$$\sum_{j=1}^{m} A_{kj} z_j \ (\leq = \geq) b_{ki}$$

$z_j \geq 0$, for $k = 1, \dots, 5$ and $i = 1, \dots, n$.

Fourth Value $= Max. V_4 = \frac{\sum_{j=1}^{m} a_{4j} w_j}{\sum_{j=1}^{m} d_{2j} w_j}$.

Subject to: (E_4)

$$\sum_{j=1}^{m} A_{kj} w_j \ (\leq = \geq) b_{ki}$$

$w_j \geq 0$, for $k = 1, \dots, 5$ and $i = 1, \dots, n$.

Fifth Value $= Max. V_5 = \frac{\sum_{j=1}^{m} a_{5j} t_j}{\sum_{j=1}^{m} d_{1j} t_j}$.

Subject to: (E_5)

$$\sum_{j=1}^{m} A_{kj} t_j \ (\leq = \geq) b_{ki}$$

$t_j \geq 0$, for $k = 1, \dots, 5$ and $i = 1, \dots, n$.

Theorem 2.3.1 [10]

Let E_k be the collection of feasible solutions to (V_k). A feasible solution $\bar{x}_k \in E_k$ is said to be the optimal solution of $(V_1, V_2, V_3, V_4, V_5)$ if there is no $x_k^* \in E_k$ such that $V_1(x^*) \geq V_1(\bar{x})$…and $V_5(x^*) \geq V_5(\bar{x})$, where $k = 1, \dots, 5$.

Theorem 2.3.2

Each optimal solution to problems $(V_1, V_2, V_3, V_4, V_5)$ is an efficient fuzzy solution for Problem 1.

Proof

Let $x_k^* \in S$ is a feasible solution to Problem 1.

Clearly, $\bar{x}_k$ are feasible solutions of $V_k, k = 1, \dots, 5$ respectively.

Now, since $\bar{x}_k$ are optimal solutions of V_k.

We have $V_k(\bar{x}_k) \geq V_k(x_k^*)$ This implies that the $Max. z(\bar{x}_k) \geq Max. z(x_k^*)$ for all feasible solutions of Problem 1

Therefore $\bar{x}_k$ it is an efficient fuzzy solution to Problem 1.

2.4 Algorithm

This section explains the stages of creating a fuzzy solution for PFLFP problems. Consider PFLFP problems.

1- Convert PFLFP problems into five LFP problems by the decomposition method.
2- Convert each LFP problem into an LP problem using Charnes and Cooper's.
3- Solve each LP problem by the simplex method.
4- Considering the optimal solution obtained from step 4, the fuzzy solution is obtained.

2.5 Flowchart

In this section, a flowchart is shown, which further explains the steps of our algorithm.

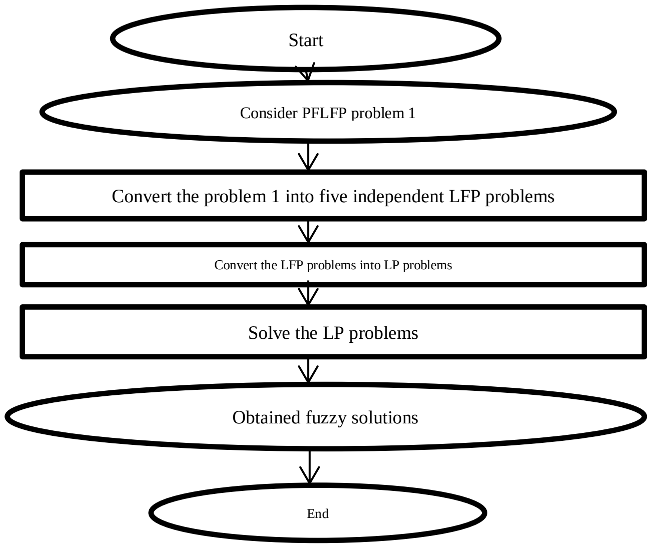

Figure 2: Flowchart of our Algorithm.

3. RESULTS AND DISCUSSIONS

3.1 Numerical Example

This section provides a detailed explanation of the proposed method. This example can be solved using several other methods by other researchers.

Consider the following PFLFP problem:

Problem 2

$$Max.z = \frac{\tilde{6}\tilde{x}_1 + \tilde{5}\tilde{x}_2 + \tilde{2}}{\tilde{3}\tilde{x}_1 + \tilde{4}\tilde{x}_2 + \tilde{5}}$$

subject to:

$$\tilde{5}\tilde{x}_1 + \tilde{6}\tilde{x}_2 \le \widetilde{10}$$
$$\tilde{3}\tilde{x}_1 + \tilde{4}\tilde{x}_2 \le \widetilde{11}$$
$$\tilde{x}_1, \tilde{x}_2 \ge \tilde{0}$$

Solution:

The following problem is taken from Problem 2 when integer coefficients are converted to PFNs:

Problem 3 $Max.z = \frac{(4,5,6,7,8)(x_1,y_1,z_1,w_1,t_1)+(3,4,5,6,7)(x_2,y_2,z_2,w_2,t_2)+(0,1,2,3,4)}{(1,2,3,4,5)(x_1,y_1,z_1,w_1,t_1)+(2,3,4,5,6)(x_2,y_2,z_2,w_2,t_2)+(1,3,5,7,9)}$

subject to:

$(2,3,5,7,8)(x_1, y_1, z_1, w_1, t_1) + (3,4,6,8,9)(x_2, y_2, z_2, w_2, t_2) \le (6,8,10,12,14)$

$(0,2,3,4,6)(x_1, y_1, z_1, w_1, t_1) + (1,2,4,6,7)(x_2, y_2, z_2, w_2, t_2) \le (7,9,11,13,15)$

where $x_1, y_1, z_1, w_1, t_1, x_2, y_2, z_2 \ge 0$

As mentioned in Section 2.4, five LFP problems can be constructed in Problem 3.

$$V_1 = \frac{4x_1 + 3x_2}{5x_1 + 6x_2 + 9}, \quad V_2 = \frac{5y_1 + 4y_2 + 1}{4y_1 + 5y_2 + 7}, \quad V_3 = \frac{6z_1 + 5z_2 + 2}{3z_1 + 4z_2 + 5}$$

Subject to: (E_1)	Subject to: (E_2)	Subject to: (E_3)
$2x_1 + 3x_2 \le 6$	$2y_1 + 3y_2 \le 6$	$2z_1 + 3z_2 \le 6$
$0x_1 + x_2 \le 7$	$0y_1 + y_2 \le 7$	$0z_1 + z_2 \le 7$
$3x_1 + 4x_2 \le 8$	$3y_1 + 4y_2 \le 8$	$3z_1 + 4z_2 \le 8$
$2x_1 + 2x_2 \le 9$	$2y_1 + 2y_2 \le 9$	$2z_1 + 2z_2 \le 9$
$5x_1 + 6x_2 \le 10$	$5y_1 + 6y_2 \le 10$	$5z_1 + 6z_2 \le 10$
$3x_1 + 4x_2 \le 11$	$3y_1 + 4y_2 \le 11$	$3z_1 + 4z_2 \le 11$
$7x_1 + 8x_2 \le 12$	$7y_1 + 8y_2 \le 12$	$7z_1 + 8z_2 \le 12$
$4x_1 + 6x_2 \le 13$	$4y_1 + 6y_2 \le 13$	$4z_1 + 6z_2 \le 13$
$8x_1 + 9x_2 \le 14$	$8y_1 + 9y_2 \le 14$	$8z_1 + 9z_2 \le 14$
$6x_1 + 7x_2 \le 15$	$6y_1 + 7y_2 \le 15$	$6z_1 + 7z_2 \le 15$
$x_1, x_2 \ge 0$	$y_1, y_2 \ge 0$	$z_1, z_2 \ge 0$

$$V_4 = \frac{7w_1 + 6w_2 + 3}{2w_1 + 3w_2 + 3}, \quad V_5 = \frac{8t_1 + 7t_2 + 4}{t_1 + 2t_2 + 1}$$

Subject to: (E_4)	Subject to: (E_5)
$2w_1 + 3w_2 \le 6$	$2t_1 + 3t_2 \le 6$
$0w_1 + w_2 \le 7$	$0t_1 + t_2 \le 7$
$3w_1 + 4w_2 \le 8$	$3t_1 + 4t_2 \le 8$
$2w_1 + 2w_2 \le 9$	$2t_1 + 2t_2 \le 9$
$5w_1 + 6w_2 \le 10$	$5t_1 + 6t_2 \le 10$
$3w_1 + 4w_2 \le 11$	$3t_1 + 4t_2 \le 11$
$7w_1 + 8w_2 \le 12$	$7t_1 + 8t_2 \le 12$
$4w_1 + 6w_2 \le 13$	$4t_1 + 6t_2 \le 13$
$8w_1 + 9w_2 \le 14$	$8t_1 + 9t_2 \le 14$
$6w_1 + 7w_2 \le 15$	$6t_1 + 7t_2 \le 15$
$w_1, w_2 \ge 0$	$t_1, t_2 \ge 0$

According to Charnes and Cooper's process, we identify an optimal solution for each of the aforementioned LFP problems.

Regarding the problem (V_1)

$Max(V_1) = 4x_1 + 3x_2$

subject to

(E_1) and $5x_1 + 6x_2 + 9 = 1$

The optimal solution is $(x_1, x_2) = (1.71427,0)$ and the value of $V_1 = 0.39024$

Then, about the solution to the problem V_2

$Max(V_2) = 5y_1 + 4y_2 + 1$

subject to

(E_2) and $4y_1 + 5y_2 + 7 = 1$

The optimal solution is $(y_1, y_2) = (1.71427,0)$ and the value of $V_2 = 0.69072$

Also, concerning the solution to the problem V_3

$Max(V_3) = 6z_1 + 5z_2 + 2$
subject to
(E_3) and $3z_1 + 4z_2 + 5 = 1$
The optimal solution is $(z_1, z_2) = (1.71427,0)$ and the value of $V_3 = 1.21126$
In the latter, referring to the solution to the problem V_4
$Max(V_4) = 7w_1 + 6w_2 + 3$
Subject to
(E_4) and $2w_1 + 3w_2 + 3 = 1$
The optimal solution is $(w_1, w_2) = (1.71428,0)$ and the value of $V_4 = 2.33333$
The last value concerning the solution to the problem $V_5$is found as a solution of the following:
$Max(V_5) = 8t_1 + 7t_2 + 4$
subject to
(E_5) and $t_1 + 2t_2 + 1 = 1$
The optimal solution is $(t_1, t_2) = (1.71428,0)$ and the value of $V_5 = 6.52631$
Therefore, the efficient fuzzy solution to the problem 7.2 is
$\tilde{x}_1 = (x_1, y_1, z_1, w_1, t_1) = (1.71427,1.71427,1.71427,1.71428,1.71428)$, and
$\tilde{x}_2 = (x_2, y_2, z_2, w_2, t_2) = (0,0,0,0,0)$.
Thus, the fuzzy value of the same problem is$(V_1, V_2, V_3, V_4, V_5) = (0.39024,0.69072,1.21126,2.33333,6.52631)$.
The same of problem 2 is solved by the proposed procedure of [10] with a suitable value of the parameter t, then the fuzzy optimal solution is
$\tilde{x}_1 = (x_1, y_1, z_1, w_1, t_1) = (0,0,0,0,1.428571)$, and
$\tilde{x}_2 = (x_2, y_2, z_2, w_2, t_2) = (0,0,0,0,0)$
Also, by the arithmetic operation of the pentagonal fuzzy number, the fuzzy value objective function is $(V_1, V_2, V_3, V_4, V_5) = (-9, -6, -3,0,14.428568)$.

Table 1. Result Comparison

Problem 2	Fuzzy optimal solution	Fuzzy objective value
Our technique	$\tilde{x}_1 = (x_1, y_1, z_1, w_1, t_1) = (1.71427,1.71427,1.71427,1.71428,1.71428)$ $\tilde{x}_2 = (x_2, y_2, z_2, w_2, t_2) = (0,0,0,0,0)$	(0.39024,0.69072,1.21126,2.33333,6.5
[10] technique	$\tilde{x}_1 = (x_1, y_1, z_1, w_1, t_1) = (0,0,0,0,1.428571)$ $\tilde{x}_2 = (x_2, y_2, z_2, w_2, t_2) = (0,0,0,0,0)$	$(-9, -6, -3,0,14.428568)$.

3.2 Discussion

The proposed method was tested on a numerical example to validate its effectiveness. The FFLFP problem was decomposed into five LFP problems, each corresponding to different values of the pentagonal fuzzy numbers. Using the Charnes and Cooper method, each LFP problem was converted into an LP problem, which was then solved using the simplex method. The optimal solutions obtained from each LP problem were aggregated to form the efficient fuzzy solution for the original FFLFP problem. The results showed that the method successfully identified the optimal fuzzy solution, demonstrating its accuracy and reliability.
The fuzzy solution provided a comprehensive range of possible outcomes, reflecting the inherent uncertainty in the problem. This analysis highlights the method's ability to handle complex, real-world optimization problems with fuzzy coefficients and decision variables. To further demonstrate the effectiveness of our technique, we solved the same problem 2 by [10] and explained the results, comparing our result and [10]result, we observe that our technique is more effective.

4. CONCLUSION

This study presents an efficient approach for solving fully fuzzy linear fractional programming problems using pentagonal fuzzy numbers and the decomposition method. By decomposing the problem into five independent linear fractional programming problems and leveraging the Charnes and Cooper method, the proposed technique effectively addresses uncertainties in optimization problems. The method's practicality and accuracy were demonstrated through a numerical example, showcasing its potential for application in various industries. Future research could extend this method to more complex fuzzy environments and explore its integration with other optimization techniques to further enhance its applicability in real-world scenarios.

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