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Action Theory of Fuzzy Bornological Groups

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ABSTRACT

This paper explores the action theory of fuzzy bornological groups. We investigate the scenario where a fuzzy bornological group $\tilde{\mathbb{G}}$ acts on a fuzzy bornological set $\tilde{\mathbb{X}}$ such that the action map $\tilde{\mathbb{G}} \times \tilde{\mathbb{X}} \to \tilde{\mathbb{X}}$ is fuzzy bounded. Initially, we establish that both left and right translation functions are fuzzy bounded maps. This motivates us to consider a new action called a fuzzy bounded action. The primary goal of studying this action is to examine the effects of the fuzzy bornological groups on the fuzzy bornological sets by partitioning the fuzzy bornological sets into orbits. Furthermore, we define an invariant subset of $\tilde{\mathbb{X}}$ under the fuzzy bounded action of $\tilde{\mathbb{G}}$. Furthermore, the intersection and the union for a collection of invariant fuzzy bounded sets are also invariant fuzzy bounded sets. Finally, we characterize the orbital structure and prove a fundamental result a fuzzy bornological group action effectively partitions the fuzzy bornological set into orbits through fuzzy bornological isomorphism.

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1. INTRODUCTION

This work aims to study the action theory of fuzzy bornological groups where a fuzzy bornological group $\tilde{\mathbb{G}}$ acts on a fuzzy bornological set $\tilde{\mathbb{X}}$ and the action map is fuzzy bounded. In the beginning, a bornology is a collection of subsets forming asystem such that for this collection to be a bornology it should cover the set and be closed under finite unions as well as being stable under heredity. To start with this study, first we must define the fuzzy bornological set and the fuzzy bornological group. A fuzzy bornological set is a bornology $\tilde{\beta}$ on a fuzzy set $\tilde{\mathbb{X}}$ that satisfies three conditions. The first condition is that $\tilde{\beta}$ covers $\tilde{\mathbb{X}}$. The second and third conditions are that $\tilde{\beta}$ is stable under finite unions and also stable under heredity. The first condition, can be satisfied in different ways, such as if the whole fuzzy set belongs to the fuzzy bornology, then $\tilde{\beta}$ covers $\tilde{\mathbb{X}}$ or $\forall x \in \tilde{\mathbb{X}}$ a fuzzy singleton $\left\{\frac{a}{x}\right\} \in \tilde{\beta}$ such that $a \in (0,1]$ or $\cup_{\tilde{B}\in\tilde{\beta}} \tilde{B} = \tilde{\mathbb{X}}$. A discrete fuzzy bornology is a power set of $\tilde{\mathbb{X}}$, a finite fuzzy bornology $\tilde{\beta}_{fin}$ is a family of finite fuzzy subsets of $\tilde{\mathbb{X}}$, a fuzzy usual bornology $\tilde{\beta}_u$ is a family of usual fuzzy bounded sets of $\tilde{\mathbb{X}}$.

A fuzzy bounded map is a map between two fuzzy bornology, see [1]. If $\tilde{\mathbb{G}}$ be a fuzzy group, then $(\tilde{\mathbb{G}}, \tilde{\beta})$ is a fuzzy bornological group if $\tilde{\beta}$ is a bornology on a fuzzy group $\tilde{\mathbb{G}}$, and the following conditions hold. The product map $g : (\tilde{\mathbb{G}}, \tilde{\beta}) \times (\tilde{\mathbb{G}}, \tilde{\beta}) \to (\tilde{\mathbb{G}}, \tilde{\beta})$ and inverse map $h : (\tilde{\mathbb{G}}, \tilde{\beta}) \to (\tilde{\mathbb{G}}, \tilde{\beta})$ are fuzzy bounded maps see [2], for more information on algebraic bornology see [3][4][5]. In this work, the action theory of fuzzy bornological groups is studied to partition a fuzzy bornological set into orbits. The main results show that the union and intersection of the collection of an invariants of $\tilde{\mathbb{X}}$ under a fuzzy subgroup in $\tilde{\mathbb{G}}$ are also invariant fuzzy subsets under the same fuzzy subgroup. In addition, we explain how a fuzzy bornological group acts on a fuzzy bornological set by fuzzy bornological isomorphism.

We show in particular, how the boundedness of the fuzzy bornological group action may be derived. At the issue of identity, the orbital fuzzy bornological fuzzy subset set is studied, and the quotient $\widetilde{\mathbb{X}}/\bar{A}$ is found to be superior to the quotient bornology. For more information on bornological structure see [6,7,8,9,10,11].

2. Action Theory of Fuzzy Bornological Groups

The methodology for this work stems from the fact that results every left translation are examples of group action and it has been proven to be a bounded map. Since translation is a group action and bounded map this motivated to study bornological group action to divide bornological sets into orbits which it is very important tools in mathematics. After that, [1] solved the problem of bounds for fuzzy sets and fuzzy groups by construct fuzzy bornological group and fuzzy bornological set. In this section, the concepts of fuzzy bornological group action are introduced, along with some fundamental facts. Here, a fuzzy bornological group acts on fuzzy bornological sets. A fuzzy bornological group $(\widetilde{\mathbb{G}}, \tilde{\beta})$ is denoted by $\widetilde{\mathbb{G}}$ and a fuzzy bornological set $(\widetilde{\mathbb{X}}, \tilde{\beta})$ is denoted by $\widetilde{\mathbb{X}}$. In this section, the research results are explained and given at the same time for a comprehensive discussion.

Definition(2.1): Let $\widetilde{\mathbb{X}}$ be a fuzzy bornology, and let $\widetilde{\mathbb{G}}$ be a fuzzy bornological group. If $\lambda: \widetilde{\mathbb{G}} \times \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$ satisfies the first and second conditions for action theory for groups and, also λ is a fuzzy bounded.Then $(\lambda, \widetilde{\mathbb{G}}, \widetilde{\mathbb{X}})$ is called a fuzzy bornological group action and $\widetilde{\mathbb{X}}$ is called a left $\widetilde{\mathbb{G}}$ - fuzzy bornological set, such that $\lambda(\widetilde{\mathbb{G}} \times \widetilde{\mathbb{X}})(y) = \sup\left\{\min\left(\widetilde{\mathbb{G}}(g), \widetilde{\mathbb{X}}(x)\right) : \lambda(g,x) = y\right\}$.

Definition (2.2): Let $g \in \widetilde{\mathbb{G}}$, then the g - translation of $(\lambda, \widetilde{\mathbb{G}}, \widetilde{\mathbb{X}})$ denoted by λ_g is the map $\lambda_g: \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$ such that $\lambda_g(x) = \lambda(g,x)$

Definition (2.3): Let $\widetilde{\mathbb{G}}$ be a fuzzy bornological group acting on fuzzy bornological set $\widetilde{\mathbb{X}}$. This action is called:

- An effective (or faithful) action if for any $g \neq e$ in $\widetilde{\mathbb{G}} \Rightarrow \lambda(g,x) \neq x$ for some x.
- A transitive action if for any two points x, y in $\widetilde{\mathbb{X}}$ there exists an element in the fuzzy group such that $\lambda(g,x) = y$.

Example(2.4): Any fuzzy bornological group $\widetilde{\mathbb{G}}$ acting on itself by the product map $\lambda: \widetilde{\mathbb{G}} \times \widetilde{\mathbb{G}} \to \widetilde{\mathbb{G}}$ such that. It is clear that the first and second conditions are satisfied and with respect to the third condition. Since, the product map is a fuzzy bounded map, then it follows that λ is fuzzy bounded action. Thus, every fuzzy bornological group $\widetilde{\mathbb{G}}$ acts on itself by the product map and this action is transitive.

Example(2.5): Let be $\widetilde{\mathbb{G}}$ be a fuzzy bornological group and $\widetilde{\mathbb{X}} = \widetilde{\mathbb{G}}$ be a fuzzy bornological set. Define the action
By $\lambda(g,h) = ghg^{-1}, \forall g,h \in \widetilde{\mathbb{G}}$. Such that

- $\lambda(e,h) = ehe^{-1} = h, \forall h, e \in \widetilde{\mathbb{G}}$;
- $\lambda(g, \lambda(g',h)) = \lambda\left(g, (g'hg'^{-1})\right) = g(g'hg'^{-1})g^{-1} = \lambda(g, g', h)$.

Then λ is a fuzzy bounded action and this action is called the conjugate action of $\widetilde{\mathbb{G}}$ on itself.

Definition (2.6): A fuzzy bornological isomorphism is a map such that this map and its inverse are fuzzy bounded . That is for every $\tilde{B} \in \tilde{\beta}$ there is $\tilde{B_1} \in \tilde{\beta}$,such that

$$\mu_{B_1}(y) = \mu_{f(\tilde{B})}(y) = sup\{\mu_{\tilde{B}}(z) : z \in f^{-1}(y) \; if \; f^{-1}(y) \neq \emptyset\} \; and$$
$$\mu_{\tilde{B}}(x) = \mu_{f^{-1}B_1}(x) = \mu_{B_1}(f(x)) \; for \; all \; x \in \mathbb{X}.$$

Theorem(2.7): The map of a fuzzy bornological group action is a fuzzy bornological isomorphism.

Proof: Since the left translation is an example of a fuzzy bounded action, then for every element $g \in \widetilde{\mathbb{G}}$ it determines a bounded translation λ_g of $\widetilde{\mathbb{X}}$ onto itself, defined by $\lambda_g(x) = \lambda(g,x)$ for each $x \in \widetilde{\mathbb{X}}$. Then by Definition (2.1) $\lambda_h \circ \lambda_g = \lambda_{hg}$ and by Definition (2.1) $\lambda_e = I_x$, the identity map of $\mathbb{X}$ onto itself. Thus $\lambda_g \circ \lambda_{g^{-1}} = \lambda_{gg^{-1}} = \lambda_e = \lambda_{g^{-1}} \circ \lambda_g$. Hence,$\lambda_g$ has a fuzzy bounded inverse map $\lambda_{g^{-1}}$, which shows that each left translation is a fuzzy bornological isomorphism.

Proposition (2.8): Let $(\lambda, \widetilde{\mathbb{G}}, \widetilde{\mathbb{X}})$ be a bornological group action such that $g \in \widetilde{\mathbb{G}}$ and $\tilde{B}$ is a fuzzy bounded subset of $\widetilde{\mathbb{X}}$, then

i) $\lambda_g(\tilde{B}) = \tilde{B}(\lambda_{g^{-1}})$.

ii) $\lambda_g(\tilde{B}^c) = 1 - \lambda_g(\tilde{B})$.

Proof: i) $\forall w \in \widetilde{\mathbb{X}}, \mu_{\lambda_g(\tilde{B})}(w) = \sup\{\mu_{(g\times\tilde{B})}(l,x) : \lambda(l,x) = w : l \in \widetilde{\mathbb{G}}, x \in \widetilde{\mathbb{X}}\}$

$= \sup\{\min(\mu_g(l), \mu_{\tilde{B}}(x)) : \lambda(l,x) = w\} = \sup\{\min(\mu_g(g), \mu_{\tilde{B}}(x)) : \lambda(g,x) = w\}$

since $\mu_g(g) \neq 0$ only when $g = l$.

$= \mu_{\tilde{B}}(x)$ where $\lambda(g,x) = w$

$= \mu_{\tilde{B}}(\lambda_{g^{-1}})(w)$

Hence (i) we have $\lambda_g(\tilde{B}) = \tilde{B}(\lambda_{g^{-1}})$ for any $\tilde{B} \subseteq \widetilde{\mathbb{X}}$ and $g \in \widetilde{\mathbb{G}}$.

ii) From (i) we have $\lambda_g(\tilde{B}) = \tilde{B}(\lambda_{g^{-1}})$ for any $\tilde{B} \subseteq \widetilde{\mathbb{X}}$ and $g \in \widetilde{\mathbb{G}}$.

And, for any $x \in \widetilde{\mathbb{X}}$, we have $\mu_{\lambda_g(\tilde{B}^c)}(x) = \mu_{\tilde{B}^c}(\lambda_{g^{-1}})(x) = 1 - \mu_{\tilde{B}}(\lambda_{g^{-1}})(x)$

Definition (2.9): Let $\widetilde{\mathbb{X}}$ be a $\widetilde{\mathbb{G}}$-fuzzy bornological set, then a fuzzy bounded subset $\tilde{B}$ of $\widetilde{\mathbb{X}}$ is called an invariant fuzzy bounded subset under $\widetilde{\mathbb{G}}$ if $\lambda(\widetilde{\mathbb{G}} \times \tilde{B}) = \tilde{B}$.

Example (2.10): Suppose $(\mathbb{Z}, \tilde{\beta}_{\text{fin}})$ is a fuzzy finite bornological group and $(\mathbb{R}, \tilde{\beta}_u)$ is a fuzzy usual bornological set, the fuzzy bornological group $\mathbb{Z}$ acts on the fuzzy bornological set $\mathbb{R}$ by $\lambda(z,r) = z + r$ for each $z \in \mathbb{Z}, r \in \mathbb{R}$ .it is clear that $\mathbb{R}$ is a $\mathbb{Z}$ - fuzzy bornological set. The set $\mathbb{Q}$ of all rational numbers as a subset of $\mathbb{R}$ is an invariant fuzzy bornological set under the action defined above of $\mathbb{Z}$, such that $\mathbb{Z} \cdot \mathbb{Q} = \{z + q : z \in \mathbb{Z}, q \in \mathbb{Q}\} = \mathbb{Q}$.

Definition (2.11): Let $\tilde{A}$ be a fuzzy bounded subset of fuzzy bornological group $\widetilde{\mathbb{G}}$ and $\tilde{B}$ a fuzzy subset of a fuzzy bornological set $\widetilde{\mathbb{X}}$. Then $\tilde{B}$ is said to be invariant under $\tilde{A}$ if $\lambda(\tilde{A} \times \tilde{B}) \subseteq \tilde{B}$ and is denoted by $\tilde{A}$ - invariant.

Proposition (2.12): Let $(\lambda, \widetilde{\mathbb{G}}, \widetilde{\mathbb{X}})$ be a fuzzy bornological group action then:
i) If $\tilde{B}$ is a fuzzy bounded subset of $\widetilde{\mathbb{X}}$, and $\tilde{A}$ is a fuzzy bounded subgroup of $\mathbb{G}$ satisfying $\tilde{A}(e) = 1$, then $\lambda(\tilde{A} \times \tilde{B}) \subseteq \tilde{B} \Leftrightarrow \lambda(\tilde{A} \times \tilde{B}) = \tilde{B}$.
ii) If $\{\tilde{B}_i\}$ is a collection of $\tilde{A}$-invariant fuzzy bounded subsets, then $\sup\tilde{B}_i$ and $\inf\tilde{B}_i$ are $\tilde{A}$-invariant.
iii) If $\lambda(g \times \tilde{B}) \subseteq \tilde{B}$, then $\lambda(g^{-1} \times \tilde{B}^c) \subseteq \tilde{B}^c$.

Proof: i) Suppose $\lambda(\tilde{A} \times \tilde{B}) \subseteq \tilde{B}$. We have for any w in $\widetilde{\mathbb{X}}$, $\mu_{\lambda(\tilde{A}\times\tilde{B})}(w) = \sup\{\mu_{(\tilde{A}\times\tilde{B})}(g,x) : \lambda(g_1,x) = w\} = \sup\{\min(\mu_{\tilde{A}}(g), \mu_{\tilde{B}}(x)) : \lambda(g,x) = w\}$

$\geq \min(\mu_{\tilde{A}}(e), \mu_{\tilde{B}}(w)$, since $\lambda(e,w) = u = \mu_{\tilde{B}}(w)$, since $\tilde{A}(e) = 1$

Thus $\lambda(\tilde{A} \times \tilde{B}) = \tilde{B}$. Consequently $\lambda(\tilde{A} \times \tilde{B}) \subseteq \tilde{B} \Leftrightarrow \lambda(\tilde{A} \times \tilde{B}) = \tilde{B}$.

ii) We have $\lambda(\tilde{A} \times \sup\tilde{B}_i) = \lambda\{\sup(\tilde{A} \times \tilde{B}_i)\} = \sup\lambda(\tilde{A} \times \tilde{B}_i) \subseteq \tilde{B}$

Similarly $\lambda(\tilde{A} \times \inf\tilde{B}_i\} = \lambda\{\inf(\tilde{A} \times \tilde{B}_i)\} \subseteq \inf\lambda(\tilde{A} \times \tilde{B}_i) \subseteq \tilde{B}$

iii) We have $\lambda(g \times \tilde{B}) \subseteq \tilde{B} \Rightarrow \lambda_g(\tilde{B}) \subseteq \tilde{B}$.

And $\lambda(g^{-1} \times \tilde{B}^c) = \lambda_{g^{-1}}(\tilde{B}^c)$.Now 1- $\tilde{B} \subseteq 1 - \lambda_g(\tilde{B}) = \tilde{B}^c\lambda_{g^{-1}}$

$\Rightarrow \tilde{B}^c \subseteq \lambda_g\tilde{B}^c$

$\Rightarrow \lambda_{g^{-1}}\tilde{B}^c \subseteq \tilde{B}^c$

$\Rightarrow \lambda(g^{-1} \times \tilde{B}^c) \subseteq \tilde{B}^c$

3. Orbit of Fuzzy Bornological Sets.

When a fuzzy bornological group $\widetilde{\mathbb{G}}$ acts on a fuzzy bornological set $\widetilde{\mathbb{X}}$, this process is called the fuzzy bornological group action as usual the effect of the fuzzy bornological group action is to partition the fuzzy bornological sets into orbits.

Definition (3.1): Let $\widetilde{\mathbb{X}}$ is a $\widetilde{\mathbb{G}}$ - fuzzy bornological set, then the orbit of x under the action is the fuzzy bounded subset $\widetilde{\mathbb{G}}_x = \lambda(\widetilde{\mathbb{G}} \times x)$.

Definition (3.2): If $(\lambda, \widetilde{\mathbb{G}}, \widetilde{\mathbb{X}})$ is a fuzzy bornological group action, let $x \in \widetilde{\mathbb{X}}$, and let $\tilde{A}$ be a fuzzy bounded subgroup of $\widetilde{\mathbb{G}}$. Then the orbit of x under $\tilde{A}$ or the $\boldsymbol{\tilde{A}}$-orbit of $\boldsymbol{x}$ is defined to be the fuzzy bounded subset $\lambda(\tilde{A} \times x)$. We will denote the orbit of x under $\tilde{A}$ by $\tilde{A}_x$, i.e. $\tilde{A}_x = \lambda(\tilde{A} \times x)$.

Proposition (3.3): Let $\widetilde{\mathbb{G}}$ be a fuzzy bornological group acting on a fuzzy bornological set $\widetilde{\mathbb{X}}$ and let $\tilde{A}_x$ be orbit of x under $\tilde{A}$, then:
(1) $\tilde{A}_x(w) = \sup\{\tilde{A}(g): \lambda(g,x) = w\}$.
(2) $\tilde{A}_x(y) = \tilde{A}_y(x)$.
(3) Let $x, y \in \widetilde{\mathbb{X}}$, such that $\tilde{A}_x(y) > 0$.Then $\tilde{A}_x(w) > 0 \Leftrightarrow \tilde{A}_y(w) > 0$ for any w in $\widetilde{\mathbb{X}}$.
(4) The $\tilde{A} -$ orbit of x is $\tilde{A} -$ invariant.

Proof: (1) We have $\mu_{\tilde{A}_x}(w) = \mu_{\lambda(\tilde{A}\times x)}(w) = \sup\{\mu_{(\tilde{A}\times x)}(g,y): \lambda(g,y) = w\} =$
$\sup\{\min(\mu_{\tilde{A}}(g), \mu_x(y): \lambda(g,y) = w\}$
$= \sup\{\mu_{\tilde{A}}(g): \lambda(g,x) = u\}$, since $\mu_x(y) \neq 0$ iff $x = y$
(2) We have $\mu_{\tilde{A}_x}(y) = \sup\{\mu_{\tilde{A}}(g): \lambda(g,x) = y\}$
$= \sup\{\mu_{\tilde{A}}(g): \lambda(g^{-1},y) = x\}$
$= \sup\{\mu_{\tilde{A}}(g^{-1}): \lambda(g^{-1},y) = x\} = \mu_{\tilde{A}_y}(x)$
(3) Given $\mu_{\tilde{A}_x}(y) > 0$. So $\exists g \in \widetilde{\mathbb{G}}$ with $\mu_{\tilde{A}}(g) > 0, \lambda(g,x) = y$. This implies
$\lambda(g^{-1},y) = x$.
Let $\mu_{\tilde{A}_x}(w) > o$, then there exists $s \in \widetilde{\mathbb{G}}$ with $\mu_{\tilde{A}}(s) > 0, \lambda(s,x) = w$. So, $\lambda(sg^{-1},y) = w$ .And $\mu_{\tilde{A}}(sg^{-1}) \geq \min(\mu_{\tilde{A}}(s), \mu_{\tilde{A}}(g^{-1})) \geq \min(\mu_{\tilde{A}}(s), \mu_{\tilde{A}}(g)) > 0$, since $\tilde{A}$ is a fuzzy subgroup $\Rightarrow \mu_{\tilde{A}_y}(w) > 0$.
Similarly $\mu_{\tilde{A}_y}(w) > 0 \Rightarrow \mu_{\tilde{A}_x}(w)$.
(4) We have for any $w \in \widetilde{\mathbb{X}}$,

$$\mu_{\lambda(\tilde{A}\times\tilde{A}_x)}(w) = \sup\{\mu_{(\tilde{A}\times\tilde{A}_x)}(g,y): \lambda(g,y) = w\}$$
$$= \sup\{\min(\mu_{\tilde{A}}(g), \mu_{\tilde{A}_x}(y)): \lambda(g,y) = w\}$$
$$= \sup[\min(\mu_{\tilde{A}}(g), \sup\{\mu_{\tilde{A}}(s): \lambda(s,x) = y\}: \lambda(g,y) = w]$$
$$= \sup_g \sup_s\{\min(\mu_{\tilde{A}}(g), \mu_{\tilde{A}}(s)): \lambda(gs,x) = w\}$$
$$\leq \sup_g \sup_s\{\mu_{\tilde{A}}(gs): \lambda(gs,x) = w\}, \text{ since } \tilde{A} \text{ is a fuzzy bounded subgroup of } \widetilde{\mathbb{G}}$$
$$= \sup\{\mu_{\tilde{A}}(r): \lambda(r,x) = w\} = \mu_{\tilde{A}_x}(w)$$ Hence $\lambda(\tilde{A} \times \tilde{A}_x) \subseteq \tilde{A}_x$.

We consider the collection of all $\tilde{A}$-orbits and define a relation on it as $\tilde{A}_x \sim \tilde{A}_y$ If $\tilde{A}_x(y) > 0$. Then, it can be easily verified that this relation is an equivalence relation, where the equivalence class of $\tilde{A}_x$ is $[\tilde{A}_x]$, such that $[\tilde{A}_x] = \{\tilde{A}_y: \tilde{A}_x(y) > 0\}$
So, $[\tilde{A}_x]$ is a fuzzy bounded subset of $\widetilde{\mathbb{X}}$ defined as $[\tilde{A}_x](y) = \tilde{A}_x(y)$. The equivalence classes with respect to this equivalence relation are the orbits of the elements of $\mathbb{X}$. Let us denote the set of all equivalence classes by $\widetilde{\mathbb{X}}/\bar{A}$, i.e. $\widetilde{\mathbb{X}}/\bar{A} = \{[\tilde{A}_x]: x \in \widetilde{\mathbb{X}}\}$. Define a map : $\widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}/\bar{A}$, given by $f(x) = [\tilde{A}_x]$. We equip $\widetilde{\mathbb{X}}/\tilde{A}$ with the quotient bornology and the set $\widetilde{\mathbb{Y}} = \widetilde{\mathbb{X}}/\tilde{A}$ is the corresponding orbit set. Let $\tilde{Y}$ carry the quotient fuzzy bornology generated by the orbital projection $f: \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}/\tilde{A}$, that means a fuzzy bounded subset $\tilde{B}_1 \subset \widetilde{\mathbb{Y}}$ is fuzzy bounded in $\tilde{Y}$ if and only if it is the image of a fuzzy bounded subset $\tilde{B}$ in $\widetilde{\mathbb{X}}$, such that $\tilde{B}_1 = f(\tilde{B})$. The orbital projection is always a fuzzy bounded map.

Theorem (3.4): If $\lambda: \widetilde{\mathbb{G}} \times \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}$ is a fuzzy bornological group action, then $f: \widetilde{\mathbb{X}} \to \widetilde{\mathbb{X}}/\tilde{A}$ given by $f(x) = [\tilde{A}_x]$ is a fuzzy bounded map.

Proof: Let $\tilde{B}$ be a fuzzy bounded subset of $\widetilde{\mathbb{X}}$. To show $f(\tilde{B})$ is a fuzzy bounded in $\widetilde{\mathbb{X}}/\tilde{A}$
Since $\widetilde{\mathbb{X}}/\tilde{A}$ has a quotient bornology with respect to f, it is sufficient to show that $f^{-1}[f(\tilde{B})]$ is a fuzzy bounded subset in $\widetilde{\mathbb{X}}$. We have $\mu_{f^{-1}[f(\tilde{B})]}(x) = \mu_{f(\tilde{B})}(f(x)) = \mu_{f(\tilde{B})}[\tilde{A}_x]$

$$= \sup\{\mu_{\tilde{B}}(y): f(y) = [\tilde{A}_x]\}$$
$$= \sup\{\mu_{\tilde{B}}(y): [\tilde{A}_y] = [\tilde{A}_x]\}$$
$$= \sup\{\mu_{\tilde{B}}(y): \tilde{A}_y \in [\tilde{A}_x]\}$$
$$= \sup\{\mu_{\tilde{B}}(y): \tilde{A}_x(y) > 0\}$$
$$= \sup\{\mu_{\tilde{B}}(y): \lambda(g,x) = y \text{ for some } g \in \widetilde{\mathbb{G}} \text{ with } \mu_{\tilde{A}}(g) > 0\}$$
$$= \sup\{\mu_{\tilde{B}}(\lambda_{g^{-1}}(x)): \lambda(g,x) = y \text{ for some } g \in \widetilde{\mathbb{G}} \text{ with } \mu_{\tilde{A}}(g) > 0\}$$
$$= \sup\left\{\mu_{\lambda_g(\tilde{B})}(x)\right): \lambda(g,x) = y \text{ for some } g \in \widetilde{\mathbb{G}} \text{ with } \bar{A}(g) > 0\right\}$$
$$= \sup \mu_{\lambda_g(\tilde{B})}(x)\Big) \text{ where } g \in \widetilde{\mathbb{G}} \text{ with } \tilde{A}(g) > 0$$

Hence $f^{-1}[f(\tilde{B})] = \sup\lambda_g\tilde{B}$. Now each λ_g is an isomorphism and $\tilde{B}$ is a fuzzy bounded and so $\lambda_g\tilde{B}$ is a fuzzy bounded. Consequently $\sup\lambda_g\tilde{B}$ is a fuzzy bounded. Hence $f^{-1}(f(\tilde{B}))$ is a fuzzy bounded. Cnsequently f is a fuzzy bounded map.

4.Conclusions

This study provides a rigorous theoretical extension of fuzzy structures by formalizing the interaction between fuzzy bornological groups $\widetilde{\mathbb{G}}$ and their respective sets $\widetilde{\mathbb{X}}$. We have established that the action of $\widetilde{\mathbb{G}}$ on $\widetilde{\mathbb{X}}$ through fuzzy bornological isomorphism induced a well-defined partition of the space into distinct orbits. This result is pivotal, as it bridges algebraic group theory with fuzzy bornological categories, offering a robust framework of investigating symmetry and boundedness in non-classical spaces. Such a partition into orbits confirms that the fundamental properties of classical group actions are preserved and enriched within the fuzzy bornological paradigm, laying the groundwork for further exploration into fuzzy dynamical systems and categorical topology.

Building upon the theoretical framework established in this study, several promising avenues for future research energy. First, we intend to investigate the topological dynamics of the induced orbits, specifically focusing on the stability and recurrence properties within the fuzzy bornological setting. Second, a natural extension involves transitioning from group actions to fuzzy bornological ring actions on modules, which would deepen the algebraic understanding of these structures. Finally, we aim to explore of universal objects within the category of fuzzy bornological space. Such investigations the gap between fuzzy categorical topology and non-classical dynamical systems.

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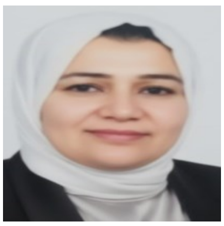

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