



ON GROUP ACTION AND COHOMOLOGY THEORY

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Article Info

Article history:

Received 08, 03, 2025

Revised 09, 15, 2025

Accepted 10, 20, 2025

Published 10, 30, 2026

Keywords:

Cohomology theory

Group action

Cohomology theory

Bornological group

group

ABSTRACT

In this work, the concept of group action and cohomology group are considered in detail. Group action is a concept in abstract algebra describes how a group acts on a set. The motivation, goal, or need for studying the action is to study the effects of the groups on the sets such that divide these sets into orbits. The motivation and goal or need for considering the cohomology group it is to study the properties of such kind of group which can be constructed from special action, such that this action can be happened when there is action from a group Q on an abelian group M and construct Q -module. After construct this kind of groups from that action and corresponding these kind of groups by each other some group homomorphisms. The techniques of group cohomology can be extended, when Q acts on a non-abelian Q -group, as a generalization of a module to non-abelian coefficients.

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1. INTRODUCTION

Group action is a fundamental concept in group theory used to understand how groups influence sets into orbits. Group action helps by studying symmetries and transformations. Group action is not just a mathematical tool, it is a means to understand and analyze the symmetries of various systems in mathematics, physics, and engineering. It provides a method to study how groups affect sets, which helps in solving and simplifying mathematical problems. In this work, we deal with cohomology theory of groups also. For more works on action theory see [1, 2].

Cohomology group has its origins in topology, starting with the works of Riemann (1857), Petit (1871), and Poincaré (1895) on the "symmetry numbers" of manifolds. The discussion was initially about an outline of finite group cohomology from the point of view of compact algebra, topology. Then the discussion became a discussion of the links between cohomology group and representation. Also, the role that cohomology group plays in the study of transformation groups is discussed [4, 5]. For more study in cohomology on bornological structure on some algebraic system see [6-8]. In this literature, we recall group action is a fundamental concept in group theory used to understand how groups influence sets into orbits. We also address the cohomology theory for groups. For more information about pure mathematics, you can see ([9, 10]). "In this article, we will denote the group by the symbol Q and the set by symbol U ".

2.Group Action: Group action is a concept in abstract algebra describing how group acts on a set.

Definition 2.1: Let $(Q, *)$ be a group, U is set "Q acts on U" if there is a map $(\lambda \text{ or } \bullet: Q \times U \rightarrow U)$. This map takes an element q of Q and element u of U and the outputs (image) is an element of U denoted by $\lambda(q, u)$ or $q \bullet u$ or qu or $q(u)$, this map to be action should satisfy these conditions:-

- (i) $e \bullet u = u \quad \forall u \in U, e$ identity in the group;
- (ii) $(q * h) \bullet u = q \bullet (h \bullet u) \quad \forall q, h \in Q, u \in U$. Then Q acts on U and U is called Q -set

Example 2.2: Let $(\mathbb{R}, +)$ be an additive group and $\mathbb{R}^2$ be a set. Define $\alpha: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as $\alpha(r, (r', r'')) = (r + r', r + r'')$, for all $r \in \mathbb{R}, (r', r'') \in \mathbb{R}^2$, we say that $\mathbb{R}$ acts on $\mathbb{R}^2$ if α satisfy these conditions :-
1- $\alpha(0, (r', r'')) = (0 + r', 0 + r'') = (r', r'') \in \mathbb{R}^2$.

$2 - \alpha(q', \alpha(q'', (r', r''))) = \alpha(q', (q'' + r', q'' + r'')) = (q' + q'' + r', q' + q'' + r'') = \alpha(q' + q'', (r', r''))$ for each $q', q'' \in \mathbb{R}$. Then α it is an action map.

Definition 2.3 (Orbits): Let Q be a group acts on a set U , then U will be partition to orbits under action, then the orbits of $u \in U$ will be denoted by $[u] = \{t \in U \mid t \sim u\} = \{t \in U \mid t = q \cdot u, q \in Q\} = \{q u \mid q \in Q\}$. The orbit of s is $[u]$ and it's subset of U .

Definition 2.4 (Fixed point): If Q is a group and U is a set. A point m is a fixed point if $q \cdot u = u$ for each $q \in Q$. (every element belong to U is fixed element if there is no effect for all element belonging to the group on this point).

Definition 2.5 (stabilizer): Q is a group acts on a set U , the stabilizer of u is subset of Q and defined as $stab(u) = \{q \in Q \mid q \cdot u = u\}$, the set of all element of the group that "fix" u . (what is the element which belong to Q such that the effect of this element on all elements belong to U is fixed?). The stabilizer for any $u \in U$, it is not empty set.

Definition 2.6 (Invariant set): U is a set, B is a subset of U , then a sub set B is called an invariant set under actions of group Q if $Q \cdot B = B$.

Definition 2.7(Kernel of action): The kernel for any action λ is denoted by $ker(\lambda) = \{q \in Q : q \cdot u = u, \forall u \in U\}$, if $ker(\lambda) = \{e\}$, then this an action is called the trivial action.

Examples 2.9: 1- Q is a group, U is a set if $Q = U$, then, $\lambda : Q \times U \rightarrow U$ given $q \in Q, u \in U$, such that $\lambda(q, u) = q \cdot u = qu$, i.e. qu is a left multiplication of Q on itself it's hold by left multiplication and this action is transitive, indeed.

i) $e \cdot u = u \quad \forall u \in U, e \in Q$.

ii) $(qh) \cdot u = q \cdot (h \cdot u) \quad \forall q, h \in Q, u \in U$.

That means, every (left) right translations is an example of group action.

2- Q is a group, U is a set and $Q = U$, thus $\lambda : Q \times U \rightarrow U$ define as $\lambda(q, u) = q u q^{-1}$

is a conjugate of U by Q which is example of group action. Every group acts on itself by conjugation. The kernel of this action is the center of Q , $Ker(\lambda) = \{q \in Q : qh = hq, \forall h \in Q\} = Z(Q)$.

3-Let $\mathbb{Z}^+ = (\mathbb{Z}^+, \cdot)$ be a group, $\mathbb{Z}$ is a set. Let $\alpha : \mathbb{Z}^+ \times \mathbb{Z} \rightarrow \mathbb{Z}$ define as $\alpha(t, n) = n + t, n \in \mathbb{Z}, t \in \mathbb{Z}^+$. Such that $\alpha(1, n) = n + \ln 1 = n \quad n \in \mathbb{Z}$.

- $\alpha(n_1, \alpha(n_2, n)) = \alpha(n_1, n + \ln n_2) = n + \ln n_1 + \ln n_2 = n + \ln n_1 n_2 = \alpha(n_1 n_2, n)$, where $n_1, n_2 \in \mathbb{Z}^+$.

4-Any group Q acts on itself by left translations, that is: $\lambda(q, x) = qx$ for any fixed arbitrary element $q \in Q$ and $x \in X = Q$. Indeed

1) $\lambda(e, x) = ex = x$ where $e \in Q$ and $x \in X$.

2) $\lambda(q_1, \lambda(q_2, x)) = \lambda(q_1, q_2 x) = q_1 q_2 x = \lambda(q_1 q_2, x)$ where $q_1, q_2 \in Q, x \in X$.

5- Q is a group, $H \subset Q$ is a subgroup. There is action $\lambda_q(q'H) = q \cdot (q'H)$. Indeed,

1) $\lambda(e, q'H) = q'H$

2) $\lambda(q_1, \lambda(q_2, q'H)) = \lambda(q_1, q_2 \cdot (q'H)) = q_1 q_2 \cdot (q'H) = \lambda(q_1 q_2, q'H)$.

3.Cohomology Theory of Group: In the following we give the sense of a short exact sequence and state the condition for which a sequence can be a short exact sequence.

Definition 3.1 : An exact sequence of groups is a collection of groups Q_n connected by collection of groups homomorphism $f_n : Q_{n-1} \rightarrow Q_n$.

Such that $Im(f_n) \subseteq ker(f_{n+1}) \Leftrightarrow f_n \circ f_{n+1} = 0$.

This is exact sequence $\left[Q_{n-2} \xrightarrow{f_{n-1}} Q_{n-1} \xrightarrow{f_n} Q_n \xrightarrow{f_{n+1}} Q_{n+1} \xrightarrow{f_{n+2}} Q_{n+2} \right]$

An exact sequence is a particular case of cochain complex and it is the chain complex.

Definition 3.2: The cochain complex (Q^*, f^*) . It consists of a sequence of abelian groups or modules, $Q^0, Q^1, Q^2, Q^3, Q^4, \dots$ connected by homeomorphisms $f_n : Q_{n-1} \rightarrow Q_n$ satisfying $f_{n-1} \circ f_n = 0$.

$$\dots \xrightarrow{f^{-1}} Q^{-1} \xrightarrow{f^0} Q^0 \xrightarrow{f^1} Q^1 \xrightarrow{f^2} Q^2 \xrightarrow{f^3} Q^3 \xrightarrow{f^4} \dots$$

Note :- A short exact sequence of groups is an exact sequence of Q consisting of five terms $1 \rightarrow N \xrightarrow{1-1} Q \xrightarrow{\text{onto}} V \rightarrow 1$. The exactness of the sequence is equivalent to three conditions.

The mapping from N (normal group) to G is one-to-one.

The mapping from Q to V (quotient group) is onto mapping.

The image of the mapping from N to Q equal to the kernel of Q .

Now, to explain cohomology group. First, let remember group action, when Q acts on X , X is called Q -set. If $X = (M, +)$ is an abelian group, $X = M$ is called $(Q - \text{modulo})$. Is group action with extra condition:

$$1 - e * m = e$$

$$2 - (qh) * m = q(hm)$$

$$3 - q * (m_1 + m_2) = q * m_1 + q * m_2.$$

Let denoted the set $C^n(Q, M) = \{f: Q^n \rightarrow M, M - \text{modulo}\}$. To have group structure, we need operation $\oplus$

Indeed, $(C^n(Q, M), \oplus)$ is abelian group of all function from Q^n to M

$$1 - \forall f, h \in C^n(Q, M) = \{f: Q^n \rightarrow M: f(q_1, q_2, \dots, q_n) = m\} \quad 1 - (f_1 \oplus f_2)(q_1, q_2, \dots, q_n) = [f(q_1, \dots, q_n) + h(q_1, \dots, q_n)] = m_1 + m_2 \in M$$

$$2 - (f_1 \oplus f_2) \oplus f_3 = f_1 \oplus (f_2 \oplus f_3) = (m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$$

$$3 - \text{Identity } (f \oplus 0)(q_1 \dots q_n) = f(q_1 \dots q_n)$$

$$4 - \text{Inverse } (f \oplus f^{-1})(q_1, \dots, q_n) = (f^{-1} \oplus f)(q_1, \dots, q_n) = 0$$

Then $(C^n(Q, M), \oplus)$ are groups (co-chain).

$$- C^1(Q, M) = [f: Q \rightarrow M, f \text{ is bijective}] \text{ 1-dim co-chain.}$$

$$- C^2(Q, M) = [f: Q^2 \rightarrow M, f \text{ is bijective}] = [f: Q \times Q \rightarrow M] \text{ 2-dim co-chain. We can related these groups}$$

$$\text{to each other's by (group homomorphism) depend on } n \text{ argument is called co-boundary map. } C^{n-1} \xrightarrow{d_n} C^n \xrightarrow{d_{n+1}} C^{n+1}$$

$$. \quad d_n: C^{n-1}(Q, M) \xrightarrow{q \cdot \text{hom}} C^n(Q, M) \quad (0 \rightarrow C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} C^2 \rightarrow \dots)$$

The co-boundary formula is given in the standard way: $(d_{n+1}f)(q_1, \dots, q_{n+1}) = q_1 \cdot f(q_2, \dots, q_{n+1}) \oplus$

$$\sum_{i=1}^n (-1)^i f(q_1, \dots, q_{i-1}, q_i, q_{i+1}, \dots, q_{n+1}) + (-1)^{n+1} f(q_1, \dots, q_n)$$

Where, $n = 0, i = 1, \dots, n+1 = 1$

$$(d_1f)(q_1) = q_1 \cdot f(q_1) + (-1)^1 f(q_1) \Rightarrow (d_1f)(q_1) = q_1 f(q_1) - f(q_1)$$

$$n = 1, i = 1, 2, \dots, n+1 = 2 \text{ (Two variables)}$$

$$(d_2f)(q_1, q_2) = q_1 f(q_2) + (-1)^1 f(q_1, q_2) + (-1)^2 f(q_1) = q_1 f(q_2) - f(q_1, q_2) + f(q_1)$$

$$n = 2, i = 1, \dots, n+1 = 3 = 1, 2, 3$$

$$(d_3f)(q_1, q_2, q_3) = q_1 f(q_2, q_3) + (-1)^1 f(q_1, q_2, q_3) + (-1)^2 f(q_1, q_2, q) + (-1)^3 f(q_1, q_2)$$

$$n = 3, i = 1, \dots, n+1 = 4 = 1, 2, 3, 4$$

$$(d_4f)(q_1, q_2, q_3, q_4) = q_1 f(q_2, q_3, q_4) + (-1)f(q_1, q_2, q_3, q_4)$$

$$+ (-1)^2 f(q_1, q_2, q_3, q_4) + (-1)^4 f(q_1, q_2, q_3)$$

Theorem 3.3: Prove that $(d_1 \circ d_0 = 0), (d_2 \circ d_1 = 0), (d_3 \circ d_2 = 0)$

Proof :- $0 \rightarrow C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} C^2$

$$[CH^0 = \frac{\ker d_1}{\text{Im } d_0} = \frac{(\ker d_1)}{0} = M^G = \text{Hom}_{Z[G]}(Z, M)]$$

$$(d_2 \circ d_1 f)(q_1, \dots, q_{n+1}) = d_2(d_1 f(q_1, \dots, q_{n+1})) = d_2((d_1 f)(q_1))$$

$$= d_2(q_1 f(q_1) - f(q_1)) = 0$$

1 - Check d_n to be hom. $d_{n+1}(f_1 \oplus f_2) = d_{n+1}(f_1) \oplus d_{n+1}(f_2)$ in $C^n \rightarrow C^{n+1}$

$$d_{n+1}(f_1 \oplus f_2)(q_1, \dots, q_{n+1}) = q \cdot (f_1 \oplus f_2)(q_2, \dots, q_{n+1}) + \dots$$

$$= q_1 \cdot f_1 + q_1 \cdot f_2 = q \cdot f_1(q_1, \dots, q_{n+1}) + q \cdot f_2(q_2, \dots, q_{n+1})$$

Then we get an exact sequence of group called n -co-chain sequence defined as follow:

$$0 \rightarrow C^0 \xrightarrow{d_1} C^1 \xrightarrow{d_2} C^2 \dots C^{n-1} \xrightarrow{d_n} C^n \dots$$

$$C^0 \left\{ \begin{array}{l} 0 - \dim \\ \text{cochain} \\ H^0 \text{ zero chomology} \end{array} \right.$$

$$C^1 \left\{ \begin{array}{l} 1 - \dim \\ \text{cochain} \\ H^1, \text{first chomology} \end{array} \right.$$

$$C^2 \left\{ \begin{array}{l} 2 - \dim \\ H^2, \text{second chomology} \end{array} \right.$$

Let remember (exact sequence of groups) is sequence of (objects- groups) and (morphisms- group homomorphism.), then n – co-chain sequence is complex if $(d_{n+1} \circ d_n = 0)$, the image of one subset of the next $\text{Im}(d_n) \subset \ker d_{n+1}$. The application of cochain is to define cohomology. $\text{coh } q = H^n(Q, M) =$

$$\frac{(\ker(d_{n+1}))}{\text{Im}(d_n)} = \frac{\text{cocycles}}{\text{coboundary}} = \frac{Z^n(Q, M)}{B^n(Q, M)} \begin{bmatrix} 0 & n = 0 \\ 1 & n \geq 1 \end{bmatrix}$$

$$H^0(Q, M) = M^G = \text{Ho } m(Z, M) = Z[GQ] \text{ group ring} = \frac{\ker(d_1)}{\text{Im}(d_0)}$$

That main is just $\ker(d_1)$ invariant element of M .

$$H^0 = \frac{\ker(d_1)}{\text{Im}(d_0)} = \ker(d_1) = M^Q = [m \in M : \forall q \in Q, q.m = m]$$

$$\left(H^1 = \frac{\ker(d_2)}{\text{Im}(d_1)} = [f: Q \rightarrow M : f(q_1, q_2) = q_1 f(q_2) + f(q_1)] \right)^1$$

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