



Stability Analysis of Food Chain System with Presence of Middle Predators Refuge and Mutual Interaction for the Top Predator with Reserved Zone

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ABSTRACT

The aim of this work is to study the local and global stability of a food chain dynamical system with a wholly dependence top predator in a reserved zone, investigating the mutualistic interaction of the top predator with both the prey and the middle predator as indemnity Holling type II functional response to obtain food in an ecologically reserved zone, where the prey and the middle predator coexist outgoing the reserved zone while the top predator easily burrows between them, the boundedness of the system is demonstrated, and the existence of the five feasible equilibrium points was proven. The sufficient conditions for the local and global stability of equilibrium points shown briefly. Persistence of the food chain system studied with support of average Lyapunov function. A numerical simulation is utilized to assure the theoretical outcomes. Finally, the research results concluded via a conclusion.

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1. INTRODUCTION

In mathematics, a dynamical system models the behavior of complex natural multidisciplinary activity covering the study of some manifestation unlike propriety, Biology, Epidemiology, Physiology, Ecology, Immunology, Genetics, Physics, etc... are some of those proprieties. This modelling become a highly appeals area of research that is contemplated by mathematicians and theoretical ecologists returns to its universal existence and importance, it helps to understand the behaviors of real-life problems, especially, the dynamic interrelationships, in food chain feeding relationship between various living things in a specific environment or habitat. Many of biological species destiny is extinction and many others are at the edge of extinction due to a lot of external effects such as over- imposition, overfishing, pollution, negligence of the habitat, etc.

In order to protect these species, appropriate measures such as restriction on harvesting, creating reserved zones. Many studies deal with such subject for instance, in [1] Rubec discusses wetland future for Iraq, the research introduce habitat for fish and wildlife for human use focused on the Chebayish and Hawizeh Marsh areas of southeastern Iraq. In such environments, the food chain represents complex interaction of more than two incorporated species in particular community [2].

On the other hand many researches focused on the study of food chains, in [3] Shireen thesis studied the dynamic of three ecological models as a dynamical system, describe food chain models, Murthy and Bahlool in [4] and [5] propose a mathematical model for a prey-wholly dependent and a prey-partially dependent predator system respectively in a reserved area for the predator, consumes the prey with Beddington-

DeAngelis type of functional response, as well as Jawad and Naji in [6] proposed a stage structure predator-prey model with a partially dependent predator and prey conservation. Many biologists experimentally spotted that the food chain system contemplate action not only simply candid and vital (comes to killing by predator) but also indirect and non-wasted (caused by refuge of predators), a "food chain refuge" is a model strain characterize how a prey population is partially protected from predators, which can vary the system's stability and behavior. Refuges can be constant or variable which settle the system by banned prey extinction, though the particular result consist on the type of refuge, consequently in [7-10].

Refuges are divided into many types according to several factors, including the geography of the place such as caves, rocks, or sand, size and type of prey (small size and their chances of escaping and hiding), the behavioral strategy of the prey, they may have the ability to fly or to create shelters and tunnels underground finally but not last [11-13]. Most researchers have discussed the dynamical behavior of food chains with several types of functional responses (is a quantitative description of the amount eaten by an average predator individual over a relatively short time period in terms of the variables affecting that amount) [14] involving many biological factors. In particular, Agarwal and Kumarb [15] proposed and studied the alternative resource effects for top predators using Holling type III model. They showed that an alternative resource has the ability to prohibit top predator annihilation. Later Xiao et al. [16] proposed and investigated a predator-prey model that associates time delay with a Holling type II and Allee impact in prey.

They observed that if the Allee impact is strong or the birth rate is low, then both species of the system are fading and hence the stability of the system is affected. Abdulqadr, D. A. & Mustafa, A. N. [17], studied global stability and Hopf-Bifurcation on delayed prey- predator model dealing with impact of fear of predators in prey, and Allee effect for predator reproduction with time delay corresponding to the gestation period on the dynamics of a predator- prey model, they discovered that the impact of the delay time between the prey predation and predator response to the predation and the predator fear on prey reproduction on the dynamic of the trajectories by modifying incorporating system with the effect of both fear and time lag.

The aim of this work is to investigate local and global stability of a food chain system consisting wholly-dependent predator with reserved zone to the prey "lower level on the food chain" proposed and analyzed, assuming that the middle predator "the center level in the food chain" behave as migration refuge. (The structure of reserve zone-refuges and free zone in a given homeland is one of these constraint), in which the predator density dependent migration of middle predator population, however the middle predator behaves as a prey relative to the top predator" the higher level" in the food chain, this refuge plays as a key role for the survival of the food chain populations, which drive the top predator interact mutually between prey and middle predator (can attack the prey and middle predator) to ensure survival, section.2 introduce the resulting system model combined of unreserved prey- reserved prey- middle and top predators in the unreserved zone, the boundedness of the system under study and existence of the certain five equilibrium points showed in the third and fourth sections respectively, local and global stability proofed in the next two sections follow 5 and 6. Persistence has an importance aspect to study in section.7, numerical analysis in section 8 rate as a key step to prove the theoretical findings above, finally in section.9 conclusion introduce the achieved results.

2. Mathematical model

Suppose that $P_1(t)$ and $P_2(t)$ represent the preys population size at the unreserved and reserved zone respectively at time t . Let $P_3(t)$, $P_4(t)$ denote the middle and top predators populations size respectively at time t . The functional response have been carried out according to Holling type II, the preys society expand logistically in the absence of predator, the middle predator expand according to the food by predation prey in the unreserved zone, depending on refuge strategy $(1-m)$, where prey may temporarily migrate or hiding, helps to survive and ensure not preyed upon by top predators,. The refuge parameter " m " confined in the interval $[0,1]$. The top predator has two main sources of food middle predator and prey and both take mutual playing the role of predator in the absence of the other. Last but not least the predation of the middle predator by the top predator is a preference for the top predator, requiring considerable effort in searching. The next system of differential equations, system (1) may be applied for meaningful the development of the food chain under the above mentioned assumption:

$$\begin{aligned} \frac{dP_1}{dt} &= rP_1 \left(1 - \frac{p_1}{k_1}\right) - \frac{\beta_1 P_1 P_3}{\alpha_1 + P_1} - \frac{\beta_2 P_1 P_4}{\alpha_2 + P_1} - \delta_1 P_1 + \delta_2 P_2 = f_1 \\ \frac{dP_2}{dt} &= sP_2 \left(1 - \frac{p_2}{k_2}\right) + \delta_1 P_1 - \delta_2 P_2 = f_2 \\ \frac{dP_3}{dt} &= \frac{\beta_3 P_1 P_3}{\alpha_1 + P_1} - \frac{\beta_4 (1-m) P_3 P_4}{\theta + \alpha_3 + (1-m) P_3} - \rho_1 P_3 = f_3 \end{aligned} \quad (1)$$

$$\frac{dP_4}{dt} = \frac{\beta_5(1-m)P_3P_4}{\theta + \alpha_3 + (1-m)P_3} + \frac{\beta_6P_1P_4}{\alpha_2 + P_1} - \rho_2P_4 = f_4$$

With $P_1 \geq 0, P_2 \geq 0, P_3 \geq 0$ and $P_4 \geq 0$ the following table (table1) describe every parameter through system (1)

Table 1. The description of system (1) parameters

Parameter	Description
r, s	The intrinsic growth rate of P_1 and P_2 respectively
k_1, k_2	The carrying capacities of the prey in the unreserved and reserved zone respectively
β_1, β_2	The maximum attack rate for the prey via middle and top predator respectively
β_3	Conversion factor of the middle predator to prey
$\alpha_1, \alpha_2, \alpha_3$	Half saturation level
m	Middle predator refuge from the top predator species ; $0 \leq m < 1$
ρ_1, ρ_2	Natural death rate coefficient of the middle and top predator respectively
β_4	Maximum attack rate for the middle predator by the top predator
β_5, β_6	Respective conversion factors of the top predator to the prey and middle predator in the food chain
δ_1, δ_2	Migration rate coefficient of the prey from unreserved area to reserved area and vise versa
θ	Search rate

All the assumed parameters in the above table (table 1) are positive

3. System Limitation

In this section we discuss boundedness of system (1), out of the following theorem:

Theorem 1. All the solution of system (1), which in $\mathbb{R}_+^4$ are uniformly bounded.

Proof. Let P_1, P_2, P_3 and P_4 be any solutions for system (1), with non-negative initial conditions such that $\mathcal{W}(t) = P_1(t) + P_2(t) + P_3(t) + P_4(t)$, by differentiation $\mathcal{W}(t)$ with respect to t we get:

$$\begin{aligned} \frac{d\mathcal{W}}{dt} = & rP_1 \left(1 - \frac{P_1}{k_1}\right) - \frac{\beta_1P_1P_3}{\alpha_1 + P_1} - \frac{\beta_2P_1P_4}{\alpha_2 + P_1} + sP_2 \left(1 - \frac{P_2}{k_2}\right) + \frac{\beta_3P_1P_3}{\alpha_1 + P_1} - \frac{\beta_4(1-m)P_3P_4}{\theta + \alpha_3 + (1-m)P_3} - \rho_1P_3 \\ & + \frac{\beta_5(1-m)P_3P_4}{\theta + \alpha_3 + (1-m)P_3} + \frac{\beta_6P_1P_4}{\alpha_2 + P_1} - \rho_2P_4 \end{aligned} \quad (2)$$

Yet simplify (2) to get:

$$\frac{d\mathcal{W}}{dt} \leq rP_1 + sP_2 - \rho_1P_3 - \rho_2P_4 - \frac{rP_1^2}{k_1} - s\frac{P_2^2}{k_2} \quad (3)$$

After several algebraic steps we have:

$$\frac{d\mathcal{W}}{dt} + \xi\mathcal{W} \leq rk_1 - sk_2 \leq rk_1 - k_2, \text{ where } \xi = \min\{r, s, \rho_1, \rho_2\}$$

Comparing the previous differential inequality with association linear differential equation and by using Gronwall inequality, to obtain:

$$\text{As } t \rightarrow \infty \text{ we obtain } 0 < \mathcal{W}(P_1, P_2, P_3, P_4) < \frac{\mu}{\xi}(1 - e^{-\xi t}) + \mathcal{W}(0)e^{-\xi t} \quad (4)$$

with $\mu = rk_1 - k_2$

$$\text{Therefore } 0 < \mathcal{W}(P_1, P_2, P_3, P_4) < \frac{\mu}{\xi} \quad (5)$$

This demonstrate that all solution of system (1) which initiate in $\mathbb{R}_+^4$ are confined in the region

$$\Omega = \{(P_1, P_2, P_3, P_4) \in \mathbb{R}_+^4 : \mathcal{W} = P_1 + P_2 + P_3 + P_4 \leq \frac{\mu}{\xi}\}$$

Lemma.1 (Basic Supposition) It will be supposed that $r > \delta_1$ and $s > \delta_2$.

Proof. It is easy to proof that if $\delta_2 = 0$, which means that the migration from the reserved zone stopped and $r < \delta_1$, then $\frac{dP_1}{dt} < 0$, then some when $\delta_1 = 0$ and $s < \delta_2$, then $\frac{dP_2}{dt} < 0$ the attention here appears in the case where $\frac{dP_1}{dt}$ and $\frac{dP_2}{dt}$ can be positive, hence the conditions that have been stated in the lemma will be taken as the main supposition.

4. Existence of the Equilibrium Points

The food chain given by system (1) has at most five equilibrium points namely $E_i = (P_{i1}, P_{i2}, P_{i3}, P_{i4})$ for $i = 0, 1, 2, 3, 4$. Here are some details of them:

- $E_0 = (P_{01}, P_{02}, P_{03}, P_{04}) = (0, 0, 0, 0)$ the annihilation equilibrium point, always exist.
- $E_1 = (P_{11}, P_{12}, P_{13}, P_{14}) = (P_{11}, P_{12}, 0, 0)$ the predators' free equilibrium point (PFEP), for $P_{12} = \frac{\delta_1}{\delta_2}P_{11} - \frac{r}{\delta_2}P_{11}(1 - \frac{P_{11}}{k_1})$ (6)

but $P_{12} > 0$, this mean

$$P_{11} > \frac{k_1}{r}(r - \delta_1) \quad (7)$$

However P_{11} is a positive root of the following third order algebraic polynomial:

$$AP_{11}^3 + BP_{11}^2 + CP_{11} + D = 0 \quad (8)$$

Where:

$$A = \frac{sr^2}{k_1\delta_1^2k_2^2} > 0$$

$$B = \frac{-2rs(r-\delta_1)}{k_1k_2\delta_2^2} < 0$$

$$C = \frac{s(r-\delta_1)^2}{k_2\delta_2^2} - \frac{r(s-\delta_2)}{k_1\delta_2} \text{ and } D = \frac{(r-\delta_1)(s-\delta_2)}{\delta_2} - \delta_1$$

By using Descartes rule of signs equation (7) has unique positive solution if the next conditions satisfied:

$$\frac{s(r-\delta_1)^2}{k_2\delta_2^2} < \frac{r(s-\delta_2)}{k_1\delta_2} \quad (9)$$

$$\frac{(r-\delta_1)(s-\delta_2)}{\delta_2} < \delta_1 \quad (10)$$

Consequently, conditions(7), (9) and (10) are sufficient for the existence of E_1 in the interior of $P_1P_2 - plane$.

iii- The third equilibrium point $E_2 = (P_{21}, P_{22}, P_{23}, 0)$ is the top predator free equilibrium point (TPFEP), in order to obtain the sufficient conditions of E_2 existence, after substitute E_2 in system (1) and completing some algebraic computations we found out that:

$$P_{21} = \frac{\rho_1\alpha_1}{\beta_3 - \rho_1} \quad (11)$$

$$P_{22} = \frac{(s-\delta_2) + \sqrt{(s-\delta_2)^2 + 4P_{21}(2s/k_2)}}{2s/k_2} \quad (12)$$

$$P_{23} = rP_{21} - \frac{P_{21}^2}{k_1} - \delta_1P_{21} + \delta_2P_{22} \quad (13)$$

$$\text{With } P_{23} > 0 \text{ to find } \delta_2P_{22} > P_{21}(\delta_1 + \frac{P_{21}}{k_1} - r) \quad (14)$$

Clearly as condition (14) hold and $\frac{\rho_1\alpha_1}{\beta_3 - \rho_1} > k_1$ we conclude that E_2 exist in the interior of $P_1P_2P_3 - plane$

iv- $E_3 = (P_{31}, P_{32}, 0, P_{34})$ is the forth equilibrium point represent the middle predator free equilibrium point (MPFEP), yet using some algebraic proses served above to obtain the following sufficient conditions of existence of E_3 in the interior of $P_1P_2P_4 - plane$:

$$P_{31} = \frac{\rho_2\alpha_2}{\beta_6 - \rho_2} > 0 \quad (15)$$

$$P_{23} = \frac{(s-\delta_2)\sqrt{(s-\delta_2)^2 + 4(s/k_2)P_{31}}}{2s/k_2} > 0 \quad (16)$$

$$P_{34} = \left(rP_{31}\left(1 - \frac{P_{31}}{k_1}\right) - \delta_1P_{31} + \delta_2P_{32}\right)\left(\frac{\alpha_2 + P_{31}}{\beta_2P_{31}}\right) > 0 \quad (17)$$

The above outcome come true as $\beta_6 > \rho_2$ and $\delta_2P_{32} > P_{31}\left(r\left(1 - \frac{P_{31}}{k_1}\right) - \delta_1\right)$.

v- Finally on account of affirmation of the interior equilibrium point (IEP), $E_4 = (P_{41}, P_{42}, P_{43}, P_{44})$ exist if the component (P_{41}, P_{42}) is positive interaction of the next isocline [10], from system(1) we conclude:

$$sP_{42}\left(1 - \frac{P_{24}}{k_2}\right) + \delta_1P_{14} - \delta_2P_{24} = 0 \quad (18)$$

$$rP_{42}\left(1 - \frac{P_{41}}{k_1}\right) - \frac{\beta_1P_{41}P_{42}}{(\alpha_1 + P_{41})} - \frac{\beta_2P_{41}P_{44}}{(\alpha_2 + P_{41})} - \delta_1P_{41} + \delta_2P_{42} = 0 \quad (19)$$

$$P_{34} = \frac{(\alpha_1 + P_{41})(\beta_4P_{44})}{\beta_3P_{41} - \rho_1(\alpha_1 + P_{41})} - \frac{(\theta + \alpha_3)}{(1-m)} \quad (20)$$

and

$$P_{44} = \frac{1}{M_1}\left(\frac{r}{\beta_1}\left(1 - \frac{P_{41}}{k_1}\right)(\alpha_1 + P_{41}) + s/\beta_1\left(1 - \frac{P_{24}}{k_2}\right)(\alpha_1 + P_{41}) + \frac{\theta + \alpha_3}{(1-m)}\right) \quad (21)$$

$$\text{Where } M_1 = \frac{\beta_2}{(\alpha_2 + P_{41})} + \frac{(\alpha_1 + P_{41})}{\beta_3P_{41} - \rho_1(\alpha_1 + P_{41})}$$

The sufficient condition of E_4 existence on the $P_1P_2P_3P_4 - plane$ come true when

$$\beta_3P_{41} - \rho_1(\alpha_1 + P_{41}) > 0 \quad (22)$$

And

$$(\alpha_1 + P_{41})(\beta_4P_{44})(1-m) > (\theta + \alpha_3)(\beta_3P_{41} - \rho_1(\alpha_1 + P_{41})) \quad (23)$$

5. Local Stability of System (1)

In this section the local stability granted conditions discussed around the above equilibrium points via the jacobian matrix of system (1)

$$D(F) = J_i = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \text{ where } F = (f_1, f_2, f_3, f_4)$$

so that:

$$\begin{aligned} a_{11} &= r \left(1 - \frac{2P_1}{k_1}\right) - \frac{\alpha_1 P_1 P_3}{(\alpha_1 + P_1)^2} - \frac{\alpha_2 P_1 P_4}{(\alpha_2 + P_1)^2} - \delta_1, & a_{21} &= \delta_1, & a_{41} &= \frac{\alpha_2 \beta_6 P_4}{(\alpha_2 + P_1)^2} \\ a_{12} &= \delta_2, & a_{22} &= s - \frac{2sP_2}{k_2} - \delta_2, & a_{42} &= 0 \\ a_{13} &= \frac{-\beta_1 P_1}{\alpha_1 + P_1}, & a_{23} &= 0, & a_{43} &= \frac{(1-m)\beta_5(\theta + \alpha_3)P_4}{(\theta + \alpha_3 + (1-m)P_3)^2} \\ a_{14} &= \frac{-\beta_2 P_1}{\alpha_2 + P_1}, & a_{24} &= 0 \\ a_{31} &= \frac{\alpha_1 \beta_3 P_3}{(\alpha_2 + P_1)^2}, & a_{32} &= 0 \\ a_{33} &= \frac{-(1-m)\beta_4(\theta + \alpha_3)P_4}{(\theta + \alpha_3 + (1-m)P_3)^2} - \rho_1 + \frac{\beta_3 P_1}{\alpha_1 + P_1}, & a_{34} &= \frac{-\beta_4(1-m)P_3}{\theta + \alpha_3 + (1-m)P_3} \\ a_{44} &= \frac{(1-m)\beta_5(\theta + \alpha_3)P_3}{\theta + \alpha_3 + (1-m)P_3} - \rho_2 + \frac{\beta_6 P_1}{\alpha_2 + P_1} \end{aligned}$$

5.1 Annihilation equilibrium point $E_0 = (0, 0, 0, 0)$

$$\text{The Jacobian matrix of system (1) at } E_0 = J_0 = \begin{bmatrix} r - \delta_1 & \delta_2 & 0 & 0 \\ \delta_1 & s - \delta_2 & 0 & 0 \\ 0 & 0 & -\rho_1 & 0 \\ 0 & 0 & 0 & -\rho_2 \end{bmatrix} \quad (24)$$

The eigenvalues of J_0 are $\vartheta_{01} = r - \delta_1$, $\vartheta_{02} = s - \delta_2$, $\vartheta_{03} = -\rho_1$ and $\vartheta_{04} = -\rho_2$, this means E_0 is a saddle point in $\mathbb{R}_+^4$, with non-empty stable and unstable manifolds.

5.2 The predators free equilibrium point (PFEP) $E_1 = (P_{11}, P_{12}, 0, 0)$

Denote J_1 is the jacobian matrix of system (1) at E_1 such that:

$$J_1 = \begin{bmatrix} r - \frac{2rP_{11}}{k_1} - \delta_1 & \delta_2 & \frac{-\beta_1 P_{11}}{\alpha_1 + P_{11}} & \frac{-\beta_2 P_{12}}{\alpha_2 + P_{11}} \\ \delta_1 & s - \frac{2sP_{12}}{k_2} - \delta_2 & 0 & 0 \\ 0 & 0 & -\rho_1 & 0 \\ 0 & 0 & 0 & -\rho_2 + \frac{\beta_6 P_{11}}{\alpha_2 + P_{11}} \end{bmatrix} \quad (25)$$

The eigenvalues of J_1 are $\vartheta_{11} = r - \frac{2rP_{11}}{k_1} - \delta_1$, $\vartheta_{12} = s - \frac{2sP_{12}}{k_2} - \delta_2$, $\vartheta_{13} = -\rho_1$ and $\vartheta_{14} = -\rho_2 + \frac{\beta_6 P_{11}}{\alpha_2 + P_{11}}$ then E_1 is locally asymptotically stable if:

$$\left(r - \frac{2rP_{11}}{k_1} - \delta_1\right) \left(s - \frac{2sP_{12}}{k_2} - \delta_2\right) - \delta_1 \delta_2 > 0 \text{ and } \frac{\beta_6 P_{11}}{\alpha_2 + P_{11}} < \rho_2 \quad (26)$$

5.3 Top predator free equilibrium point $E_2 = (P_{21}, P_{22}, P_{23}, 0)$

For the top predator free equilibrium point E_2 the variation matrix J_2 written as following:

$$J_2 = \begin{bmatrix} r \left(1 - \frac{2P_{21}}{k_1}\right) - \frac{\alpha_1 P_{21} P_{23}}{(\alpha_1 + P_{21})^2} - \delta_1 & \delta_2 & \frac{-\beta_1 P_{21}}{\alpha_1 + P_{21}} & \frac{-\beta_2 P_{21}}{\alpha_2 + P_{21}} \\ \delta_1 & s - \frac{2sP_{22}}{k_2} - \delta_2 & 0 & 0 \\ \frac{\alpha_1 \beta_3 P_{23}}{(\alpha_2 + P_{21})^2} & 0 & -\rho_1 + \frac{\beta_3 P_{21}}{\alpha_1 + P_{21}} & \frac{-\beta_4(1-m)P_{23}}{\theta + \alpha_3 + (1-m)P_{23}} \\ 0 & 0 & 0 & \frac{(1-m)\beta_5 P_{23}}{\theta + \alpha_3 + (1-m)P_{23}} - \rho_2 + \frac{\beta_6 P_{21}}{\alpha_2 + P_{21}} \end{bmatrix} \quad (27)$$

With eigenvalues $\vartheta_{21} = r \left(1 - \frac{2P_{21}}{k_1}\right) - \frac{\alpha_1 P_{21} P_{23}}{(\alpha_1 + P_{21})^2} - \delta_1$, $\vartheta_{22} = s - \frac{2sP_{22}}{k_2} - \delta_2$, $\vartheta_{23} = -\rho_1 + \frac{\beta_3 P_{21}}{\alpha_1 + P_{21}}$ and $\vartheta_{24} = \frac{(1-m)\beta_5(\theta + \alpha_3)P_{23}}{\theta + \alpha_3 + (1-m)P_{23}} - \rho_2 + \frac{\beta_6 P_{21}}{\alpha_2 + P_{21}}$

In order to achieve E_2 local stability, the next conditions should come true:

$$\left(r \left(1 - \frac{2P_{21}}{k_1}\right) - \frac{\alpha_1 P_{21} P_{23}}{(\alpha_1 + P_{21})^2} - \delta_1\right) \left(s - \frac{2sP_{22}}{k_2} - \delta_2\right) - \delta_1 \delta_2 > 0 \quad (28)$$

$$\frac{\beta_3 P_{21}}{\alpha_1 + P_{21}} < \rho_1 \quad (29)$$

$$\frac{(1-m)\beta_5(\theta+\alpha_3)P_{23}}{\theta+\alpha_3+(1-m)P_{23}} + \frac{\beta_6 P_{21}}{\alpha_2 + P_{21}} < \rho_2 \quad (30)$$

Other wise E_2 is a saddle point.

5.4 Middle predator free equilibrium point $E_3 = (P_{31}, P_{32}, 0, P_{34})$

The variation matrix of system(1) at $E_3 = (P_{31}, P_{32}, 0, P_{34})$ is given by:

$$J_3 = \begin{bmatrix} r \left(1 - \frac{2P_{31}}{k_1}\right) - \frac{\alpha_2 P_{31} P_{34}}{(\alpha_2 + P_{31})^2} - \delta_1 & \delta_2 & \frac{-\beta_1 P_{31}}{\alpha_1 + P_{31}} & \frac{-\beta_2 P_{31}}{\alpha_2 + P_{31}} \\ \delta_1 & s - \frac{2sP_{32}}{k_2} - \delta_2 & 0 & 0 \\ 0 & 0 & \frac{-(1-m)\beta_4 P_{34}}{(\theta + \alpha_3)} - \rho_1 + \frac{\beta_3 P_{31}}{\alpha_1 + P_{31}} & 0 \\ \frac{\alpha_2 \beta_6 P_{34}}{(\alpha_2 + P_{31})^2} & 0 & \frac{(1-m)\beta_5 P_{34}}{(\theta + \alpha_3)} & -\rho_2 + \frac{\beta_6 P_{31}}{\alpha_2 + P_{31}} \end{bmatrix} \quad (31)$$

It is easy to verify that E_3 is local stable equilibrium point, if the eigenvalues of J_3 satisfy the following relations:

$$\vartheta_{31} \cdot \vartheta_{32} > 0 \text{ and } \vartheta_{31} + \vartheta_{32} < 0 \quad (32)$$

$$\vartheta_{33} \cdot \vartheta_{34} > 0 \text{ and } \vartheta_{33} + \vartheta_{34} < 0 \quad (33)$$

where : $\vartheta_{31} = r \left(1 - \frac{2P_{31}}{k_1}\right) - \frac{\alpha_2 P_{31} P_{34}}{(\alpha_2 + P_{31})^2} - \delta_1$, $\vartheta_{32} = s - \frac{2sP_{32}}{k_2} - \delta_2$, $\vartheta_{33} = \frac{-(1-m)\beta_4 P_{34}}{(\theta + \alpha_3)} - \rho_1 + \frac{\beta_3 P_{31}}{\alpha_1 + P_{31}}$ and $\vartheta_{34} = -\rho_2 + \frac{\beta_6 P_{31}}{\alpha_2 + P_{31}}$ are the eigenvalues of (31)

Otherwise E_3 is a saddle point.

5.5 Local stability of interior equilibrium point $E_4 = (P_{41}, P_{42}, P_{43}, P_{44})$

The jacobian matrix of system (1) at E_4 written as:

$$J_4 = [c_{ij}]_{3 \times 3} \quad (34)$$

where:-

$$\begin{aligned} c_{11} &= r \left(1 - \frac{2P_{41}}{k_1}\right) - \frac{\alpha_1 P_{41} P_{43}}{(\alpha_1 + P_{41})^2} - \frac{\alpha_2 P_{41} P_{44}}{(\alpha_2 + P_{41})^2} - \delta_1, & c_{21} &= \delta_1 \\ c_{12} &= \delta_2, & c_{22} &= s - \frac{2sP_{42}}{k_2} - \delta_2 \\ c_{13} &= \frac{-\beta_1 P_{41}}{\alpha_1 + P_{41}}, & c_{23} &= 0 \\ c_{14} &= \frac{-\beta_2 P_{41}}{\alpha_2 + P_{41}}, & c_{24} &= 0 \\ c_{31} &= \frac{\alpha_1 \beta_3 P_{43}}{(\alpha_2 + P_{41})^2}, & c_{32} &= 0 \\ c_{33} &= \frac{-(1-m)\beta_4(\theta+\alpha_3)P_{44}}{(\theta+\alpha_3+(1-m)P_{43})^2} - \rho_1 + \frac{\beta_3 P_{41}}{\alpha_1 + P_{41}}, & c_{34} &= \frac{-\beta_4(1-m)P_{43}}{\theta+\alpha_3+(1-m)P_{43}} \\ c_{44} &= \frac{(1-m)\beta_5(\theta+\alpha_3)P_{43}}{\theta+\alpha_3+(1-m)P_{43}} - \rho_2 + \frac{\beta_6 P_{41}}{\alpha_2 + P_{41}}, & c_{41} &= \frac{\alpha_2 \beta_6 P_{44}}{(\alpha_2 + P_{41})^2} \\ c_{42} &= 0, & c_{43} &= \frac{(1-m)\beta_5(\theta+\alpha_3)P_{44}}{(\theta+\alpha_3+(1-m)P_{43})^2} \end{aligned}$$

The characteristic equation [35] of the jacobian matrix determined as:

$$\vartheta^4 + A_1 \vartheta^3 + A_2 \vartheta^2 + A_3 \vartheta + A_4 = 0 \quad (35)$$

Where:

$$A_1 = -(c_{11} + c_{22} + c_{33} + c_{44})$$

$$A_2 = (c_{22}c_{11} - c_{21}c_{12}) + (c_{33}c_{11} - c_{31}c_{13}) + (c_{44}c_{11} - c_{41}c_{14}) + (c_{44}c_{33} - c_{43}c_{34}) + c_{33}c_{22} + c_{44}c_{22}$$

$$A_3 = -c_{11}c_{22}(c_{33} + c_{44}) - c_{33}c_{44}(c_{11} + c_{22}) + c_{33}(c_{12}c_{21} + c_{12}c_{44} + c_{14}c_{41}) + c_{22}(c_{13}c_{31} - c_{14}c_{41} + c_{34}c_{43}) + c_{22}(c_{11}c_{43} - c_{13}c_{41}) + c_{31}(c_{13}c_{44} - c_{14}c_{43})$$

$$A_4 = c_{21}c_{12}c_{43}c_{34} - c_{44}(c_{22}c_{13}c_{21} + c_{21}c_{12}c_{33})$$

Now by using the Roth-Hurwitz criteria [19, 20], the equilibrium point E_4 is locally asymptotically stable if the following relation satisfied:

$$A_1 > 0, A_3 > 0, A_4 > 0 \text{ and } \Delta > 0 \text{ where } \Delta = A_1A_2A_3 - (A_3^2 + A_1^2A_4) \quad (36)$$

The relation (36) come true if the following conditions in the blow description satisfied

$$c_{11} < 0, c_{22} < 0, c_{33} < 0 \text{ and } c_{44} < 0 \quad (37)$$

$$c_{12}c_{21} + c_{12}c_{44} + c_{14}c_{41} > 0 \quad (38)$$

$$c_{11}c_{43} - c_{13}c_{41} > 0 \quad (39)$$

$$A_2 > \frac{A_3}{A_1} + \frac{A_1A_4}{A_3} > 0 \quad (40)$$

Otherwise E_4 is a saddle equilibrium point.

6. Global Stability analysis of System (1)

In this section the global stability analysis on the equilibrium points of system (1) described and demonstrate according to the sufficient conditions of local stability described in the above section.

Theorem 2. Assume that the equilibrium point $E_1 = (P_{11}, P_{12}, 0, 0)$ is locally asymptotically stable, then it is globally asymptotically stable provided that $\beta_2\beta_3\beta_5 < \beta_1\beta_4\beta_6$ (28)

Proof.

Consider the following function:

$$R_1 = K_1 \left(P_1 - P_{11} - P_{11} \ln \frac{P_1}{P_{11}} \right) + K_2 \left(P_2 - P_{12} - P_{12} \ln \frac{P_2}{P_{12}} \right) + K_3 P_3 + K_4 P_4 \quad (29)$$

Clearly $R_1: \mathbb{R}_+^4 \rightarrow \mathbb{R}$, where K_1, K_2, K_3 and K_4 are positive constant to be determined, now by differentiating R_1 with respect to time t for system (1) gives

$$\frac{dR_1}{dt} = H_1 \left(\frac{P_1 - P_{11}}{P_1} \right) \frac{dP_1}{dt} + H_2 \left(\frac{P_2 - P_{12}}{P_2} \right) \frac{dP_2}{dt} + H_3 \frac{dP_3}{dt} + H_4 \frac{dP_4}{dt} \quad (30)$$

Now substituting $\frac{dP_1}{dt}, \frac{dP_2}{dt}, \frac{dP_3}{dt}$ and $\frac{dP_4}{dt}$ from system (1) leads to

$$\begin{aligned} \frac{dR_1}{dt} = & H_1(P_1 - P_{11}) \left[r + \frac{-r}{K_1}(P_1 - P_{11}) - \frac{\beta_1 P_3}{\alpha_1 + (P_1 - P_{11})} - \frac{\beta_2 P_4}{\alpha_2 + (P_1 - P_{11})} - \delta_1 + \delta_2 \left(\frac{P_2 P_{11} - P_1 P_{12}}{P_1 P_{11}} \right) \right] + H_2(P_2 - P_{12}) \left[s + \frac{-s}{K_2}(P_2 - P_{12}) - \delta_1 + \delta_2 \left(\frac{P_1 P_{12} - P_2 P_{11}}{P_2 P_{12}} \right) \right] + H_3 P_3 \left[\frac{\beta_3(P_1 - P_{11})}{\alpha_1 + (P_1 - P_{11})} - \frac{\beta_4(1-m)P_4}{\theta + \alpha_3 + (1-m)P_3} - \rho_1 \right] + \\ & H_4 P_4 \left[\frac{\beta_5(1-m)P_3}{\theta + \alpha_3 + (1-m)P_3} + \frac{\beta_6(P_1 - P_{11})}{\alpha_2 + (P_1 - P_{11})} - \rho_2 \right] \quad (31) \end{aligned}$$

Choosing the constants H_1, H_2, H_3 and H_4 tactfully as $H_1 = 1, H_2 = \frac{\delta_2 P_{12}}{\delta_1 P_1}, H_3 = \frac{\beta_1}{\beta_3}$ and $H_4 = \frac{\beta_2}{\beta_6}$

However, after some algebraic simplification we get:

$$\frac{dR_1}{dt} < \frac{-r}{K_1}(P_1 - P_{11})^2 - \frac{s\delta_2 P_{12}}{K_2 \delta_1 P_1}(P_2 - P_{12})^2 - \frac{\delta_2}{P_1 P_2 P_{11}}(P_1 P_{12} - P_2 P_{11})^2 \quad (31)$$

Hence $\frac{dR_1}{dt} < 0$, as condition (28) holds, consequently E_1 is a globally asymptotically stable equilibrium point.

Theorem 3. Suppose that the equilibrium point $E_2 = (P_{21}, P_{22}, P_{23}, 0)$ is locally asymptotically stable, then it is globally asymptotically stable provided that: $\beta_2\beta_3\beta_5 < \beta_1\beta_4\beta_6$. (32)

Proof.

Let us consider a definite positive function for the equilibrium point E_2 defined by:

$$R_2 = G_1 \left(P_1 - P_{21} - P_{21} \ln \frac{P_1}{P_{21}} \right) + G_2 \left(P_2 - P_{22} - P_{22} \ln \frac{P_2}{P_{22}} \right) + G_3 \left(P_3 - P_{23} - P_{23} \ln \frac{P_3}{P_{23}} \right) + G_4 P_4 \quad (33)$$

Where G_1, G_2, G_3 and G_4 are real constant to be suitably chosen, farther since

$$\frac{dR_2}{dt} = G_1 \left(\frac{P_1 - P_{21}}{P_1} \right) \frac{dP_1}{dt} + G_2 \left(\frac{P_2 - P_{22}}{P_2} \right) \frac{dP_2}{dt} + G_3 \left(\frac{P_3 - P_{23}}{P_3} \right) \frac{dP_3}{dt} + G_4 \frac{dP_4}{dt} \quad (34)$$

Then substituting the values of $\frac{dP_1}{dt}, \frac{dP_2}{dt}, \frac{dP_3}{dt}$ and $\frac{dP_4}{dt}$ from system (1) to obtain that:

$$\begin{aligned} \frac{dR_2}{dt} = & G_1(P_1 - P_{21}) \left[r \left(1 - \frac{(P_1 - P_{21})}{k_1} \right) - \frac{\beta_1(P_3 - P_{23})}{\alpha_1 + (P_1 - P_{21})} - \frac{\beta_2 P_4}{\alpha_2 + (P_1 - P_{21})} - \delta_1 + \delta_2 \left(\frac{P_2 P_{21} - P_1 P_{22}}{P_1 P_{21}} \right) \right] + G_2(P_2 - P_{22}) \left[s \left(1 - \frac{(P_2 - P_{22})}{k_2} \right) - \delta_2 + \delta_1 \left(\frac{P_1 P_{22} - P_2 P_{21}}{P_2 P_{22}} \right) \right] + G_3(P_3 - P_{23}) \left[\frac{\beta_3(P_1 - P_{21})}{\alpha_1 + (P_1 - P_{21})} - \frac{\beta_4(1-m)P_4}{\theta + \alpha_3 + (1-m)(P_3 - P_{23})} - \rho_1 \right] + \\ & G_4 P_4 \left[\frac{\beta_5(1-m)(P_3 - P_{23})}{\theta + \alpha_3 + (1-m)(P_3 - P_{23})} + \frac{\beta_6(P_1 - P_{21})}{\alpha_2 + (P_1 - P_{21})} - \rho_2 \right] \quad (35) \end{aligned}$$

Simplifying the above equation (35) after choosing suitable values for the constants G_1, G_2, G_3 and G_4 as $G_1 = 1, G_2 = \frac{\delta_2 P_{21}}{\delta_1 P_1}, G_3 = \frac{\beta_1}{\beta_3}$ and $G_4 = \frac{\beta_2}{\beta_6}$ further mathematical computations the result is

$$\frac{dR_2}{dt} \leq -\left(\frac{r}{k_1}\right)(P_1 - P_{21})^2 - \frac{s\delta_2 P_{22}}{\delta_1 P_{21} k_2}(P_2 - P_{22})^2 - \frac{\delta_2^2}{P_1 P_{21} P_{22}}(P_1 P_{22} - P_2 P_{21})^2 \quad (36)$$

At last we finally end up with $\frac{dR_2}{dt} < 0$

Therefore the coexistence equilibrium point E_2 satisfied as the condition (32) hold.

The proof of global stability of E_2 is done in a similar way.

Theorem 4. Suppose that the local stability of the interior equilibrium point $E_4 = (P_{41}, P_{42}, P_{43}, P_{44})$ satisfied then it is global asymptotically stable when.

$$\beta_2 \beta_3 \beta_5 < \beta_1 \beta_4 \beta_6 \quad (37)$$

Proof. Consider the positive definite function

$$R_4 = L_1 \left(P_1 - P_{41} - P_{41} \ln \frac{P_1}{P_{41}} \right) + L_2 \left(P_2 - P_{42} - P_{42} \ln \frac{P_2}{P_{42}} \right) + L_3 \left(P_3 - P_{43} - P_{43} \ln \frac{P_3}{P_{43}} \right) + L_4 \left(P_4 - P_{44} - P_{44} \ln \frac{P_4}{P_{44}} \right) \quad (38)$$

where L_1, L_2, L_3 and L_4 are positive constant to be determined, now by differentiating R_1 with respect to time t for system (1) gives

$$\frac{dR_4}{dt} = L_1 \left(\frac{P_1 - P_{41}}{P_1} \right) \frac{dP_1}{dt} + L_2 \left(\frac{P_2 - P_{42}}{P_2} \right) \frac{dP_2}{dt} + L_3 \left(\frac{P_3 - P_{43}}{P_3} \right) \frac{dP_3}{dt} + L_4 \left(\frac{P_4 - P_{44}}{P_4} \right) \frac{dP_4}{dt} \quad (39)$$

$$\begin{aligned} \frac{dR_4}{dt} = & L_1 (P_1 - P_{41}) \left[r \left(1 - \frac{(P_1 - P_{41})}{k_1} \right) - \frac{\beta_1 (P_3 - P_{43})}{\alpha_1 + (P_1 - P_{41})} - \frac{\beta_2 (P_4 - P_{44})}{\alpha_2 + (P_1 - P_{41})} - \delta_1 + \delta_2 \left(\frac{P_2 P_{41} - P_1 P_{42}}{P_1 P_{41}} \right) \right] \\ & + L_2 (P_2 - P_{42}) \left[s \left(1 - \frac{(P_2 - P_{42})}{k_2} \right) + \delta_1 \left(\frac{P_1 P_{42} - P_2 P_{41}}{P_2 P_{42}} \right) - \delta_2 \right] + L_3 (P_3 - P_{43}) \left[\frac{\beta_3 (P_1 - P_{41})}{\alpha_1 + (P_1 - P_{41})} - \frac{\beta_4 (1-m)(P_4 - P_{44})}{\theta + \alpha_3 + (1-m)(P_3 - P_{43})} - \rho_1 \right] \\ & + L_4 (P_4 - P_{44}) \left[\frac{\beta_5 (1-m)(P_3 - P_{43})}{\theta + \alpha_3 + (1-m)(P_3 - P_{43})} + \frac{\beta_6 (P_1 - P_{41})}{\alpha_2 + (P_1 - P_{41})} - \rho_2 \right] \end{aligned} \quad (40)$$

By choosing positive constant $L_1 = 1, L_2 = \frac{\delta_2 P_{42}}{\delta_1 P_{41}}, L_3 = \frac{\beta_1}{\beta_3}$ and $L_4 = \frac{\beta_1}{\beta_6}$, we get

$$\frac{dR_4}{dt} = -\left(\frac{r}{k_1}\right)(P_1 - P_{41})^2 - \frac{s\delta_2 P_{42}}{K_2 \delta_1 P_{41}}(P_2 - P_{42})^2 - \frac{\delta_2}{P_1 P_2 P_{41}}(P_1 P_{42} - P_2 P_{41})^2 \quad (41)$$

Therefore $\frac{dR_4}{dt} \leq 0$ as condition (37) satisfied, and E_4 is globally asymptotically stable.

7. Persistence

The persistence of system (1) studied in this section using Gard technique [21], clearly system (1) has four main systems:

$$\frac{dP_1}{dt} = h_1(P_1, P_2) = rP_1 \left(1 - \frac{P_1}{k_1} \right) - \delta_1 P_1 + \delta_2 P_2 \quad (42)$$

$$\frac{dP_2}{dt} = h_2(P_1, P_2) = sP_2 \left(1 - \frac{P_2}{k_2} \right) - \delta_2 P_2 + \delta_1 P_1$$

$$\frac{dP_1}{dt} = h_3(P_1, P_3) = rP_1 \left(1 - \frac{P_1}{k_1} \right) - \delta_1 P_1 - \frac{\beta_1 P_1 P_3}{\alpha_1 + P_1} \quad (43)$$

$$\frac{dP_3}{dt} = h_4(P_1, P_2) = -\rho_1 P_3 - \frac{\beta_1 P_1 P_3}{\alpha_1 + P_1}$$

$$\frac{dP_1}{dt} = h_5(P_1, P_4) = rP_1 \left(1 - \frac{P_1}{k_1} \right) - \delta_1 P_1 - \frac{\beta_2 P_1 P_4}{\alpha_2 + P_1} \quad (44)$$

$$\frac{dP_4}{dt} = h_6(P_1, P_4) = -\rho_2 P_4 - \frac{\beta_2 P_1 P_4}{\alpha_2 + P_1}$$

$$\frac{dP_3}{dt} = h_7(P_3, P_4) = -\rho_1 P_3 - \frac{\beta_4 (1-m) P_3 P_4}{\alpha_3 + \theta + (1-m) P_3} \quad (45)$$

$$\frac{dP_4}{dt} = h_8(P_3, P_4) = -\rho_2 P_4 - \frac{\beta_5 (1-m) P_3 P_4}{\alpha_3 + \theta + (1-m) P_3}$$

The subsystems in (42), (43), (44) and (45) continuous in the interior of $P_1 P_2$ -plane, $P_1 P_3$ -plane, $P_1 P_4$ -plane

And $P_3 P_4$ -plane. Now we will define the function $T_1(P_2, P_1) = \frac{1}{P_1 P_2}$ is continuous function in $\mathbb{R}_2^+$ of

$P_1 P_2$ -plane, next define the following quantity

$$\Delta(P_1, P_2) = \frac{\partial(h_1 T_1)}{\partial P_1} + \frac{\partial(h_2 T_1)}{\partial P_2} \quad (46)$$

$$h_1 T_1 = \frac{1}{P_2} \left(r \left(1 - \frac{P_1}{k_1} \right) - \delta_1 + \delta_2 \frac{P_2}{P_1} \right) \quad (47)$$

$$h_2 T_1 = \frac{1}{P_1} \left(s \left(1 - \frac{P_2}{k_2} \right) - \delta_2 + \delta_1 \frac{P_1}{P_2} \right) \quad (48)$$

we gain

$$\Delta(P_1, P_2) = \left(\frac{-r}{k_1 P_2} + \frac{-\delta_2}{P_1^2} \right) + \left(\frac{-s}{k_2 P_1} + \frac{-\delta_1}{P_2^2} \right) \neq 0 \quad (49)$$

According to Dulac- Bendixon criterion [22] the sign of the quantity $\Delta(P_1, P_2)$ does not change and it is not identically zero, therefore there is no closed curve in the interior of $\mathbb{R}_2^+$ of the $P_1 P_2$ -plane. In addition since the sub system (42) is bounded and has unique positive equilibrium point in the interior of $\mathbb{R}_2^+$ of $P_1 P_2$ -plane that is given by (P_{11}, P_{12}) , this conclude that according to Poincare-Bendixon theorem [22] it is globally asymptotically stable whenever it exist and locally asymptotically stable.

Similarly by using the C^1 function in the interior of $\mathbb{R}_2^+$ of the $P_1 P_3$, $P_1 P_4$ and $P_3 P_4$ planes which are defined as $\frac{1}{P_1 P_3}$, $\frac{1}{P_1 P_4}$ and $\frac{1}{P_3 P_4}$ respectively, it is straightforward to verify there is no periodicity in the interior of $\mathbb{R}_2^+$ of their planes, and the positive equilibrium point of every subsystem (P_{21}, P_{22}) , (P_{31}, P_{32}) and (P_{41}, P_{42}) is globally asymptotically stable wherever it exist and locally asymptotically stable. Thus sufficient and necessary condition for the uniform persistence of system (1) are presented in the following theorem.

Theorem 5. System (1) is uniformly persistent.

Proof. Define $\sigma(P_1, P_2, P_3, P_4) = P_1^{v_1} P_2^{v_2} P_3^{v_3} P_4^{v_4}$ as average Lyapunov function with positive constant v_i ; $i = 1, 2, 3, 4$, it is clear $\sigma(P_1, P_2, P_3, P_4)$ is a C^1 positive definite function and if $P_1 \rightarrow 0$ or $P_2 \rightarrow 0$ or $P_3 \rightarrow 0$ or $P_4 \rightarrow 0$ then $\sigma(P_1, P_2, P_3, P_4) \rightarrow 0$, thus we get that :

$$\mathfrak{D}(P_1, P_2, P_3, P_4) = \frac{\sigma'(P_1, P_2, P_3, P_4)}{\sigma(P_1, P_2, P_3, P_4)} = v_1 \left[r \left(1 - \frac{P_1}{k_1} \right) - \delta_1 - \frac{\beta_1 P_3}{\alpha_1 + P_1} - \frac{\beta_2 P_4}{\alpha_2 + P_1} + \frac{\delta_2}{P_1} \right] + v_2 \left[s \left(1 - \frac{P_2}{k_2} \right) - \delta_2 + \frac{\delta_1}{P_2} \right] + v_3 \left[\frac{\beta_3 P_1}{\alpha_1 + P_1} - \frac{\beta_4 (1-m) P_4}{\theta + \alpha_3 + (1-m) P_3} - \rho_1 \right] + v_4 \left[\frac{\beta_5 (1-m) P_3}{\theta + \alpha_3 + (1-m) P_3} + \frac{\beta_6 P_1}{\alpha_2 + P_1} - \rho_2 \right] \quad (50)$$

Now according to Gard technique, the proof is followed and the system is uniformly persisting when $\mathfrak{D}(P_1, P_2, P_3, P_4) > 0$ at each point of the equilibria, since $\mathfrak{D}(E_0) = v_1(r - \delta_1) - v_2\delta_2 - v_3\rho_1 - v_4\rho_2$ $\mathfrak{D}(E_0) > 0$ as we chose positive constant, therefore v_1 is sufficiently large with respect to the other constant and $r - \delta_1 > 0$.

And $\mathfrak{D}(E_1) = v_1 \left[r \left(1 - \frac{P_1}{k_1} \right) - \delta_1 + \frac{\delta_2}{P_1} \right] + v_2 \left[s \left(1 - \frac{P_2}{k_2} \right) - \delta_2 + \frac{\delta_1}{P_2} \right] - v_3\rho_1 - v_4\rho_2$ so

$\mathfrak{D}(E_1) > 0$ if $\left[r \left(1 - \frac{P_1}{k_1} \right) - \delta_1 + \frac{\delta_2}{P_1} \right] > 0$ this mean $r \left(1 - \frac{P_1}{k_1} \right) + \frac{\delta_2}{P_1} > \delta_1$ and $\left[s \left(1 - \frac{P_2}{k_2} \right) - \delta_2 + \frac{\delta_1}{P_2} \right] > 0$ to get

$s \left(1 - \frac{P_2}{k_2} \right) + \frac{\delta_1}{P_2} > \delta_2$ with suitable selection of ρ_1, ρ_2 (positive constant).

Similarly it is easy to prove that $\mathfrak{D}(E_2) > 0$ if the following conditions satisfied:

$$r \left(1 - \frac{P_1}{k_1} \right) + \frac{\delta_2}{P_1} > \delta_1 + \frac{\beta_1 P_3}{\alpha_1 + P_1}, s \left(1 - \frac{P_2}{k_2} \right) + \frac{\delta_1}{P_2} > \delta_2, \frac{\beta_3 P_1}{\alpha_1 + P_1} > \rho_1 \text{ and } \frac{\beta_5 (1-m) P_3}{\theta + \alpha_3 + (1-m) P_3} + \frac{\beta_6 P_1}{\alpha_2 + P_1} > \rho_2$$

And $\mathfrak{D}(E_3) > 0$ when the next conditions come true:

$$r \left(1 - \frac{P_1}{k_1} \right) + \frac{\delta_2}{P_1} > \delta_1 + \frac{\beta_2 P_4}{\alpha_2 + P_1}, s \left(1 - \frac{P_2}{k_2} \right) + \frac{\delta_1}{P_2} > \delta_2, \frac{\beta_3 P_1}{\alpha_1 + P_1} > \frac{\beta_4 (1-m) P_4}{\theta + \alpha_3 + (1-m) P_3} + \rho_1 \text{ and } \frac{\beta_6 P_1}{\alpha_2 + P_1} > \rho_2$$

Finally $\mathfrak{D}(E^*) > 0$ as $\left(1 - \frac{P_1}{k_1} \right) + \frac{\delta_2}{P_1} > \delta_1 + \frac{\beta_1 P_3}{\alpha_1 + P_1} + \frac{\beta_2 P_4}{\alpha_2 + P_1}$, $s \left(1 - \frac{P_2}{k_2} \right) + \frac{\delta_1}{P_2} > \delta_2$, $\frac{\beta_3 P_1}{\alpha_1 + P_1} > \frac{\beta_4 (1-m) P_4}{\theta + \alpha_3 + (1-m) P_3} + \rho_1$ and $\frac{\beta_5 (1-m) P_3}{\theta + \alpha_3 + (1-m) P_3} + \frac{\beta_6 P_1}{\alpha_2 + P_1} > \rho_2$

So that every necessities of Gard technique are satisfied.

8. Numerical Analysis

In this section, the global dynamic of system (1) is studied numerically, to confirm the acquired analytical results and recognize the action of variation in the parameters on the dynamic of the studied system. The following assumed set of parameters (51) utilized global stability of system (1) in simulating the numerical analysis, applying MATLAB program with sixth order Runge -Kutta method [23] to solve system (1),

$$\theta = 0.3, r = 0.8, s = 0.7, m = 0.7, k_1 = 0.2, k_2 = 0.15, \alpha_1 = 0.4, \alpha_2 = 0.5, \alpha_3 = 0.2, \delta_1 = 0.8, \delta_2 = 0.8, \rho_1 = 0.1, \rho_2 = 0.0525, \beta_1 = 0.5, \beta_2 = 0.3, \beta_3 = 0.6, \beta_4 = 0.9, \beta_5 = 0.14, \beta_6 = 0.18 \quad (51)$$

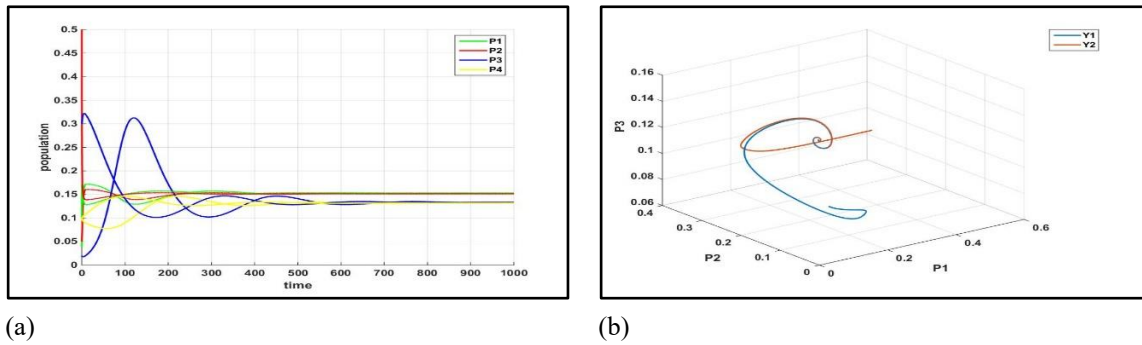


Figure 1. Dynamical behavior of system (1) for the given values of the parameters in (51), at two different initial points $Y1 = (0.04, 0.05, 0.02, 0.1)$, $Y2 = (0.1, 0.5, 0.3, 0.1)$, (a) time series trajectories of population, (b) 3-D solutions of system (1) approaching global equilibrium point

Clearly Figure 1 Explain that system (1) is globally asymptotically stable to the interior equilibrium point $(0.1529, 0.1515, 0.1332, 0.1318)$ starting from two different initial points $Y1, Y2$ which confirm the analytical results in theorem 4, Figure 1-(a) represent the time series solution of system (1), while trajectories in Figure 1-(b) denote the 3-D phase portrait which approaches globally the interior equilibrium point.

The next step is come by simulate the varying of certain parameters at every turn so as to explain its impact on the dynamics and stability of system (1), first of all the beginning is with refuge parameter m , in Figure 2(a-b) the reduction of refuge parameter from $m = 0.6$ to $m = 0.4$ had been shown, till $m = 0.2$ in Figure 2-(e-f) occur the influence on the time series trajectories from (asymptotic stable to periodic) and approaching different attractors. However this means that system (1) is sensitive at the refuge parameter m .

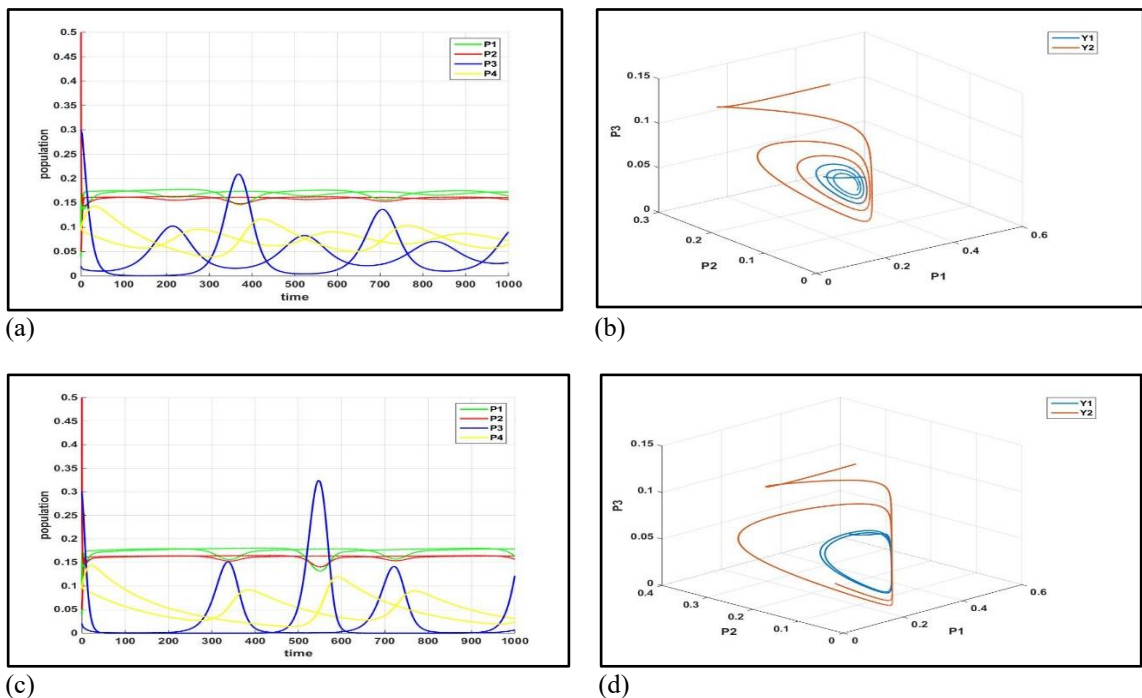


Figure 2. Dynamical behavior turnover of the trajectories approaching asymptotically stable equilibrium point in figures (a-b) according to periodic trajectories verses periodic attractors in (a, b - c, d), for $m = 0.4$ and $m = 0.1$ respectively as shown in: (a) periodic trajectories of system (1) population at $m = 0.4$, (b) 3-D periodic attractor of system (1) at $m = 0.4$, (c) periodic trajectories of system (1) population at $m = 0.1$, (d) periodic attractor of system (1) at $m = 0.1$.

Next, in Figure 3. (a) recognize the convergence of time series solution of system (1) to the predators free equilibrium point $E_3 = (P_{11}, P_{12}, 0.0) = (0.1862, 0.06823, 0.0)$ as $\delta_2 = 0.1$ (the migration parameter of the reserved prey (fixing the rest data set of parameters in (42)), while Figure 3. (b) correspond the 3-D solution set of the system (1) for initial points $Y1 = (0.04, 0.05, 0.02, 0.1)$, $Y2 = (0.1, 0.5, 0.3, 0.1)$.

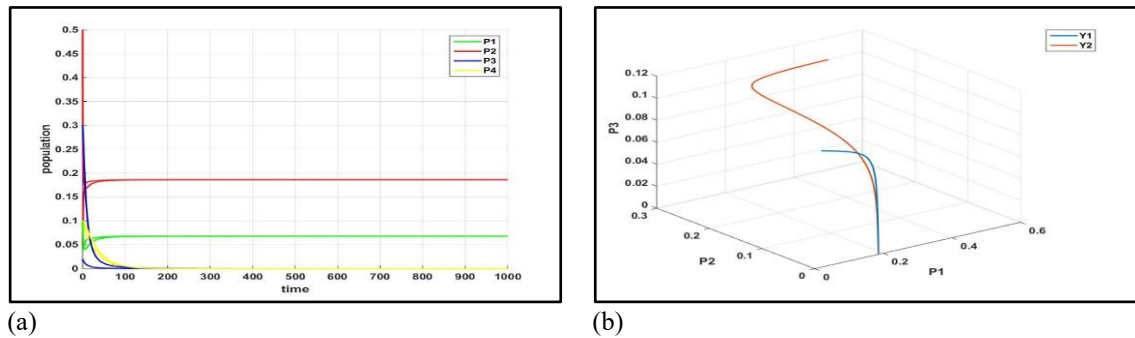


Figure 3. Convergence to the predator free equilibrium point as $\delta_2 = 0.1$, starting from the initial points $Y1 = (0.04, 0.05, 0.02, 0.1)$, $Y2 = (0.1, 0.5, 0.3, 0.1)$. descend to the predators free equilibrium point $(0.1862, 0.06823, 0, 0)$ covered in: (a) time series solution of system(1) for $\delta_2 = 0.1$, (b) 3-D trajectory reaching the predators free equilibrium point.

The intervals when certain parameters lie in according to the data set in (51) change the dynamic of system (1) around the interior equilibrium point, for instance as $\beta_6 > 0.22$ the interior equilibrium point E_4 draw near the middle predator free equilibrium point $E_3 = (0.1479, 0.149, 0, 0.1982)$, also for $\alpha_1 \leq 0.16$ the interior equilibrium point E_4 approach the top predator free equilibrium point $E_2 = (0.03221, 0.06946, 0.6137, 0)$ as shown in Figure 4. (a), and Figure 4. (b) respectively.

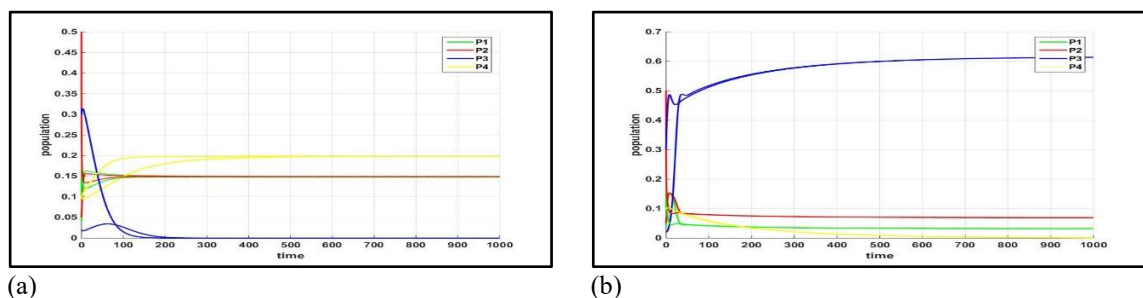


Figure 4. time series trajectories of system (1) (a) approach the middle predator free equilibrium point $E_3 = (0.1479, 0.149, 0, 0.1982)$, (b) approach the top predator free equilibrium point $(0.03221, 0.06946, 0.6137, 0)$, determined in: (a) system(1) time series trajectory for $\beta_6 = 0.23$, (b) system(1) time series trajectory for $\alpha_1 = 0.16$

9. Results and Discussion

In this work, stability analysis of a food chain model personifies middle predator refuge against top predator and mutualistic interaction by the top predator amidst middle predator and prey, the food is wasted according to Lotka-Volterra functional response. Furthermore, the middle predator grows logistically by the favorite food at the prey. The top predator behaves mutually as a skilled predator and preys upon both species the middle predator and prey in the unreserved zone. All the properties of the solution of system (1) are scanned. It is spotted that system (1) has five nonnegative equilibrium points. The conditions of local stability for each equilibrium point are structured. The global dynamics, whenever exists, is inspected too. On the other hand, the persistence conditions of the food chain model are established. It is noticed that the system has a wide range of persistence due to the existence of the middle predator refuge that is depending on both prey and predator species as well as mutuality of the top predator for each of the prey and middle predator species. In addition, a numerical simulation is used to confirm our obtained results. It is observed that the system has global stability around the interior equilibrium point, and the effect of the refuge parameter on the time series from asymptotically stable to periodic and the other parameters included in the system on the dynamic of the interior equilibrium point E_4 turning to predators free equilibrium point E_1 and middle predator free equilibrium point E_3 or top predator equilibrium point E_2 , this mentioned that the existence of the middle predator refuge depends on both prey and top predator species is a stabilizing factor on the dynamics of food chain model and expands the range of the parameters at which the system persists.

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