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Fuzzy Bornological Group

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Abstract

The aim of this work is to construct a new structure which is called fuzzy group bornology to solve the problem of bounded for fuzzy group. Furthermore, we explain that the intersection of collections of fuzzy bornological groups is a fuzzy bornological group. Also, we gave the sufficient condition to show the image of a fuzzy bornological group is fuzzy bornological group and showed there exist a fuzzy bornological isomorphism between two fuzzy bornological groups. Finally, we studied a homomorphism between fuzzy bornological groups with some main properties and some results and show that a fuzzy bornological group is homogenous space.

Keywords: Bornological Set, Fuzzy Set, Bounded Set, Bounded Map, Fuzzy Group.

الزمرة البورنولوجية الضبابية

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الخلاصة

الهدف من هذا العمل هو بناء هيكل جديد يسمى الزمرة البورنولوجية الضبابية لحل مشكلة التقيد للزمرة الضبابية. علاوة على ذلك، فاننا نوضح تقاطع عائلة من الزمر البورنولوجية الضبابية هي زمرة بورنولوجية ضبابية. ايضا، اعطينا شرطا

Academic Science Journal

كافيا لإظهار صورة زمرة بورنولوجية ضبابية هي زمرة بورنولوجية ضبابية واظهرنا وجود تماثل بورنولوجي ضبابي بين زمرتين بورنولوجيتين ضبابيتين. اخيرا، قمنا بدراسة التشابه بين الزمر البورنولوجية الضبابية مع بعض الخصائص الرئيسية وبعض النتائج وبينا ان الزمرة البورنولوجية الضبابية هي فضاء متجانس.

الكلمات المفتاحية: المجموعة البورنولوجية، المجموعة الضبابية، المجموعة المقيدة، الدالة المقيدة، الزمرة الضبابية.

Introduction

Bornology was developed by many researchers, see [1] and the fundamental monograph by [2]. The effect of bornology is to solve the problem of bounded for any set or space in general way see ([3], [4]). The theory of fuzzy set is generalization of convention set theory that was introduced by Zadeh [5]. Fuzzy set emerges due to uncertainty in real life situations and provides us an opportunity to express vague concepts in natural language as meaningful one. Here we solve the problem of boundedness for the fuzzy group. In 2001 [6], they studied the fuzzy bornological on vector space which gives rise to the notion of fuzzy vector space. In this present paper the concept of fuzzy bornological group (FBG) is introduced and some constructions for this new structure. We also discuss some of its important types and study some properties of bounded map if it is injection and surjection fuzzy bounded map. We denote $\mathbb{G}$ for fuzzy bornological group. By next part we explain the concept of the fuzzy set.

1. Fuzzy Set

Definition 1.1: Suppose X be a nonempty set and $I = [0,1]$ the unit interval. In X a fuzzy set $\bar{B}$ is characterized by a membership function $\mu_{\bar{B}}$ which associated with each point $x \in X$ its grade of membership $\mu_{\bar{B}}(x) \in I$. Which is written as $\bar{B} = \{(x, \mu_{\bar{B}}(x))\}$ or $\{\frac{\mu_{\bar{B}(x)}}{x}\}$. The collection of all fuzzy sets in X is denoted by I^X [7].

We can just use $\bar{B}$ for $\mu_{\bar{B}}$ since it is characteristic of the fuzzy set $\bar{B}$.

Definition 1.2: Suppose X and Y be two nonempty sets, $f: X \to Y$ be a mapping and $\bar{A}$ be a fuzzy subset of X. Then $f(\bar{A})$ is a fuzzy subset of Y defined by

Academic Science Journal

$f(\bar{A})(y) = \{ \sup_{x \in f^{-1}(y)} \bar{A}(x) f^{-1}(y) \neq \emptyset \}$ for all $y \in Y$, where

$f^{-1}(y) = \{x: f(x) = y\}$. If $\bar{B}$ is a fuzzy subset of Y, then the fuzzy subset $f^{-1}(\bar{B})$ of X is defined by $f^{-1}(\bar{B})(x) = \bar{B}\big(f(x)\big)$ for all $x \in X$ [7].

Definition 1.3 : Let $\bar{A}$ and $\bar{B}$ two fuzzy sets which are defined on X, then $\bar{A} \vee \bar{B}$ is a fuzzy union of $\bar{A}$ and $\bar{B}$ can be described as, $\bar{A} \cup \bar{B} = \max\,[\bar{A}(x), \bar{B}(x)]$ for all $x \in X$. Also, $\bar{A} \wedge \bar{B}$ is a fuzzy intersection of $\bar{A}$ and $\bar{B}$ can be defined as $\bar{A} \cap \bar{B} = \min[\bar{A}(x), \bar{B}(x)], \forall\, x \in X$. Also, $\bar{A} \subseteq \bar{B} \Leftrightarrow \bar{A}\,(x) \leq \bar{B}(\mathrm{x})$ for all $x \in X$ [7].

Definition 1.4 Fuzzy Bornological Set :

Let $\mathbb{X}$ be a fuzzy set. A bornology on $\mathbb{X}$ is a family of fuzzy bounded subsets of $\mathbb{X}$, such that:

(1) $\bigvee\{\bar{B} : \bar{B} \in \bar{\beta}\} = \mathbb{X} \in \bar{\beta}$;

(2) If $\bar{B} \in \bar{\beta}, \bar{C} \subseteq \bar{B}$, then $\bar{C} \in \bar{\beta}$;

(3) If $\bar{B}_1, \bar{B}_2 \in \bar{\beta}$ then $\bar{B}_1 \cup \bar{B}_2 \in \bar{\beta}$.

$\bar{\beta}$ is called ***a fuzzy bornology***, further it is denoted by FBS. The members $\mathbb{X}, \bar{\emptyset}$ are constant fuzzy bounded sets on $\mathbb{X}$ [8].

2. Fuzzy Bornological Groups

Since a bornology is benefit in bounded problem, we resolve in this part the problem of bounded for fuzzy group by construct a fuzzy bornological group.

Definition 2.1 (Fuzzy Bornological Group)

Let $\mathbb{G}$ be a fuzzy group. We say that $(\mathbb{G}, \bar{\beta})$ is a ***fuzzy bornological group*** if $\bar{\beta}$ is a fuzzy bornology on $\mathbb{G}$, and the following conditions holds:

i) The product map $g: (\mathbb{G}, \bar{\beta}) \times (\mathbb{G}, \bar{\beta}) \rightarrow (\mathbb{G}, \bar{\beta})$ is fuzzy bounded;

ii) The inverse map $h: (\mathbb{G}, \bar{\beta}) \rightarrow (\mathbb{G}, \bar{\beta})$ is fuzzy bounded.

Example 2.2:

i) Let $\mathbb{G}$ be a fuzzy group and $\bar{\beta}$ is a discrete fuzzy bornology on $\mathbb{G}$. Then $(\mathbb{G},\bar{\beta})$ is a fuzzy bornological group.

ii) Fuzzy bornology on general linear group.

Let $\mathbb{G}L(2,2) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}, +\right)$ be a general linear group which it is the set of fuzzy matrix elements and additive operation. We can define a fuzzy bornology $\bar{\beta}$ on this fuzzy group which it is the collection of all finite fuzzy subsets.

To prove $\mathbb{G}L(2,2)$ with finite fuzzy bornology $\bar{\beta}$ is a fuzzy bornological group $(\mathbb{G}L(2,2),\bar{\beta})$.

We have to show that the product and inverse maps are fuzzy bounded.

1) $\alpha: (\mathbb{G}L(2,2),\bar{\beta}) \times (\mathbb{G}L(2,2),\bar{\beta}) \to (\mathbb{G}L(2,2),\bar{\beta})$.

Let $\bar{M}_1, \bar{M}_2$ are two bounded fuzzy sets (finite sets) in $(\mathbb{G}L(2,2),\bar{\beta})$.

We must prove that $\alpha(\bar{M}_1 \times \bar{M}_2)$ is fuzzy bounded.

$$\alpha(\bar{M}_1 \times \bar{M}_2) = \bar{M}_1 + \bar{M}_2 = \{m_1 + m_2, m_1 \in \mathbb{G}L(2,2), m_2 \in \mathbb{G}L(2,2)\}$$

$$= \{m_1 + m_2 : m_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } m_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, a_{11}, a_{12}, a_{21} a_{22}, b_{11}, b_{12}, b_{21}, b_{22} \in [0,1]\}$$

$$=\{\begin{bmatrix} \max\{a_{11}, b_{11}\} & \max\{a_{12}, b_{12}\} \\ \max\{a_{21}, b_{21}\} & \max\{a_{22}, b_{22}\} \end{bmatrix}\} \subset (\mathbb{G}L(2,2),\bar{\beta})\}.$$

Thus the product map is fuzzy bounded.

2) $\alpha^{-1}: \left(\mathbb{G}L(2,2),\bar{\beta}\right) \to \left(\mathbb{G}L(2,2),\bar{\beta}\right)$.

Let $\bar{M} \in (\mathbb{G}L(2,2),\beta)$.

Then $\bar{M} = \left\{m : m = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right\}$ which is finite fuzzy set.

Then

$$M^{-1} = -M = \{-m: -m = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix}\} \subset (\mathbb{G}L(2,2), \bar{\beta}).$$

So, α^{-1} is a fuzzy bounded map.

Thus $\left(\mathbb{G}L(2,2), \bar{\beta}\right)$ is a fuzzy bornological group.

Definition 2.3: Let $\mathbb{G}$ be a fuzzy group and $\bar{\beta}$ a fuzzy bornology on $\mathbb{G}$. Let $\bar{A}, \bar{B}$ be two fuzzy sets in $\mathbb{G}$. We define $\bar{A}\,\bar{B}(x) = \sup_{x=x_1x_2} \min(\bar{A}\,(x_1), \bar{B}(x_2))$ and $\bar{B}^{-1}(x) = \bar{B}(x^{-1})$ for $x \in \mathbb{G}$.

Definition 2.4 (Fuzzy Subgroup) : Let $\bar{A} \in I^G$ be a fuzzy set such that G be a group, then $\bar{A}$ is called ***a fuzzy subgroup*** of G iff :

i) $\bar{A}\,(xy) \geq \min(\bar{A}\,(x), \bar{A}\,(y))$ for all $x, y \in G$;
ii) $\bar{A}\,(x^{-1}) \geq \bar{A}\,(x)$ for all $x \in G$ [9].

Remark 2.5:

i) $\bar{A}$ is a fuzzy subgroup iff $\bar{A}\,(xy^{-1}) \geq \min(\bar{A}\,(x), \bar{A}\,(y))$.
ii) If $\bar{A}$ is a fuzzy subgroup in a group G and e is the identity of G, then $\bar{A}\,(e) \geq \bar{A}\,(x)$ and $\bar{A}\,(x) = \bar{A}\,(x^{-1})$ for all $x \in G$ [9].

Remark 2.6:

i) If H is a subset of a group G, then $\bar{H}$ is a fuzzy subgroup of G.
ii) All constant fuzzy sets of a group G are fuzzy subgroups of G [10].

Definition 2.7: A group homomorphism between two fuzzy bornological groups is fuzzy bounded if the image for every fuzzy bounded set is fuzzy bounded set.

Theorem 2.8: Let $\bar{A}$ is a fuzzy subgroup in G and H be a group. Suppose that f is a homomorphism of G into H. Then $f(\bar{A})$ is fuzzy subgroup in H [9].

Theorem 2.9: Let H be a group and $\bar{B}$ is fuzzy subgroup in H. Let f be a homomorphism of G into H then $f^{-1}(\bar{B})$ is fuzzy subgroup in G [9].

Intersection a family of fuzzy bornological groups is a fuzzy bornological group.

Theorem 2.10: The intersection of a family of fuzzy bornological groups is a fuzzy bornological group.

Proof:

Let $\bar{\beta} = \bigcap_{i=1}^{n} \bar{\beta}_i$, it is clear that $\bar{\beta}$ is a fuzzy bornology. We must prove that it is a fuzzy bornological group.

Let $\bar{A}, \bar{B} \in \bar{\beta}$, then $\bar{A}$, $\bar{B} \in \bar{\beta}_i$, where i= $1, \dots, n$ and $x, y, z \in \mathbb{G}$.

The product map: $\alpha(\bar{A} \times \bar{B})(z) = \vee \{(\bar{A} \wedge \bar{B})\ (x, y): \alpha(x, y) = z\}$

$$= \vee\{(\bar{A}\,(x) \wedge \bar{B}(y)): xy = z\} = \bar{A}\,\bar{B}(xy) \subseteq (\mathbb{G}, \bar{\beta}).$$

Which it is a fuzzy bounded set.

The inverse map: $\alpha(\bar{B})(z) = \vee \{\bar{B}(x) : \alpha(x) = z\}$

$$= \vee \{\bar{B}(x): x^{-1} = z\}$$

$$= \vee \{\bar{B}(x) : x = z^{-1}\}$$

$$= \bar{B}(z^{-1}) = \bar{B}^{-1}(z) \subseteq \left(\mathbb{G}, \bar{\beta}\right).$$

Which it is a fuzzy bounded.

Thus $\bar{\beta}$ is a fuzzy bornological group.

We give the sufficient condition to explain the image of a fuzzy bornological group is a fuzzy bornological group.

Theorem 2.11: Let $(\mathbb{G}, \bar{\beta})$ be a fuzzy bornological group and let $f: \mathbb{G} \to \mathbb{H}$ be a homomorphism of fuzzy groups, then $(\mathbb{H}, f(\bar{\beta}))$ is fuzzy bornological group if f is surjective.

Proof:

Let $\bar{B}_1, \bar{B}_2 \in f(\bar{\beta})$ then $\exists\ \bar{A}_1, \bar{A}_2 \in \bar{\beta}$ such that $f(\bar{A}_1) = \bar{B}_1, f(\bar{A}_2) = \bar{B}_2$.

Let $u = f(x), v = f(y)$ for some $x, y \in \mathbb{G}, z \in \mathbb{H}$

The product map:

$$\alpha(\bar{B}_1 \times \bar{B}_2)(z) = \vee \{(\bar{B}_1 \wedge \bar{B}_2)(u, v) : \alpha(u, v) = z\}$$

$$= \vee \{\bar{B}_1(u) \wedge \bar{B}_2(v) : uv = z\}$$

$$= \vee \left\{\bar{B}_1(f(x)) \wedge \bar{B}_2\big(f(y)\big) : f(xy) = z\right\}$$

$$= \bar{B}_1\bar{B}_2(z) \subseteq \Big(\mathbb{H}, f\big(\bar{\beta}\big)\Big).$$

Which it is a fuzzy bounded set.

The inverse map:

$$\alpha(\bar{B}_1)(z) = \vee \{\bar{B}_1(u) : \alpha(u) = z\}$$

$$= \vee \{\bar{B}_1(u) : u^{-1} = z\}$$

$$= \{\bar{B}_1(u) : u = z^{-1}\} = \bar{B}_1(z^{-1}) = {\bar{B}_1}^{-1}(z) \subseteq \Big(\mathbb{H}, f\big(\bar{\beta}\big)\Big).$$

Which it is a fuzzy bounded, then $(\mathbb{H}, f(\bar{\beta}))$ is a fuzzy bornological group.

Theorem 2.12: Let $(\mathbb{G}, \bar{\beta})$ be a fuzzy bornology group and let $f : \mathbb{H} \to \mathbb{G}$ be a homomorphism of fuzzy groups then $(\mathbb{H}, f^{-1}(\bar{\beta}))$ is a fuzzy bornological group.

Proof:

In fact if $\bar{B}_1, \bar{B}_2 \in f^{-1}(\bar{\beta})$ then $\bar{B}_1 \subseteq f^{-1}(\bar{A}_1)$ and $\bar{B}_2 \subseteq f^{-1}(\bar{A}_2)$ for $\bar{A}_1, \bar{A}_2 \in \bar{\beta}$..

Let $x, y \in \mathbb{H}$, then the product map:

$$\alpha(f^{-1}(\bar{A}_1)) \times (f^{-1}(\bar{A}_2))\,(z) = \vee \left\{\big(f^{-1}(\bar{A}_1)\big)(x) \wedge (f^{-1}(\bar{A}_2))(y) : \alpha(x, y) = z\right\}$$

$$=\vee\{(f^{-1}(\bar{A}_1)(x)\wedge\left(f^{-1}(\bar{A}_2)\right)(y):xy=z\}$$

$$=(f^{-1}(\bar{A}_1))(f^{-1}(\bar{A}_2))(z)\subseteq(\mathbb{G},f^{-1}(\bar{\beta})).$$

Which it is a fuzzy bounded set.

And the inverse map:

$$\alpha(f^{-1}(\bar{A}_1))(z)=\vee\{\left(f^{-1}(\bar{A}_1)\right)(x):\alpha(x)=z\}$$

$$=\vee\{(f^{-1}(\bar{A}_1))(x):x^{-1}=z\}$$

$$=(f(\bar{A}_1)^{-1})^{-1}(z)\subseteq(\mathbb{G},f^{-1}(\bar{\beta})).$$

Which it is a fuzzy bounded set, then $(\mathbb{G},f^{-1}(\beta))$ is a fuzzy bornological group.

Theorem 2.13: Let $\bar{\beta}_1$ and $\bar{\beta}_2$ be any two discrete fuzzy bornological groups on the fuzzy groups $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively.Then every group homomorphism $f:\mathbb{G}_1\to\mathbb{G}_2$ is a fuzzy bounded map. But the convers in general not true.

Proof:

Since $\bar{\beta}_1$ and $\bar{\beta}_2$ are discrete fuzzy bornological groups we have for every $\bar{A}\in\bar{\beta}_1$, $f(\bar{A})\in\bar{\beta}_2$.

Hence f is a fuzzy bounded map from $\mathbb{G}_1$ to $\mathbb{G}_2$.

In the following example we explain that a fuzzy bounded map need not in general be a group homomorphism.

Example 2.14: Let $G_1=\{1,-1\}$ is a group and $G_2=\{1,C,C^2\}$ be the group of cube roots of unity under the usual multiplication with discrete fuzzy bornological groups $\bar{\beta}_1$ and $\bar{\beta}_2$ on $\mathbb{G}_1$ and $\mathbb{G}_2$ respectively.

Now define $f:(\mathbb{G}_1,\bar{\beta}_1)\to(\mathbb{G}_2,\bar{\beta}_2)$ by $f(1)=1$ and $f(-1)=C$.

Academic Science Journal

Since $\bar{\beta}_1$ and $\bar{\beta}_2$ are discrete fuzzy bornological groups, clearly for every $\bar{A} \in \bar{\beta}_1$, $f(\bar{A}) \in \bar{\beta}_2$.

Now we will prove that if $\bar{A}$ is fuzzy bounded subgroup of $\mathbb{G}_1$ in $\bar{\beta}_1$ then $f(\bar{A})$ is a fuzzy bounded subgroup of $\mathbb{G}_2$ in $\bar{\beta}_2$.

Case (1): If $\bar{A} = \overline{\emptyset}_{\mathbb{G}_1}$ or $\bar{A} = \mathbb{G}_1$, then clearly $f(\bar{A})$ is a fuzzy subgroup of $\mathbb{G}_2$

such that $f\left(\overline{\emptyset}_{\mathbb{G}_1}\right)(y) = \sup\left\{\overline{\emptyset}_{\mathbb{G}_1}(x) : f(x) = y\right\} = 0$(as $x \in \mathbb{G}_1) = \overline{\emptyset}_{\mathbb{G}_2}$

and $\quad f(\mathbb{G}_1)(y) = \sup\{\mathbb{G}_1(x): f(x) = y\} = 1$ (as $x \in \mathbb{G}_1) = \mathbb{G}_2$.

Case (2):

$$\bar{A}(x) = \begin{cases} t_1 & if \;\; x = 1 \\ t_2 & i\,f \;\; x = -1 \end{cases}$$

where, $1 \geq t_1 \geq t_2 \geq 0$.

Now for every $f(x) \in G_2$,

$$f(\bar{A})(f(x) = \begin{cases} t_1 & if \;\; f(x) = 1 \\ t_2 & if \;\; f(x) = C, \end{cases}$$

where, $1 \geq t_1 \geq t_2 \geq 0$.

Hence $f(\bar{A})$ is a fuzzy subgroup of $\mathbb{G}_2$ in $\bar{\beta}_2$.

So, f is a fuzzy bounded map from $\mathbb{G}_1$ to $\mathbb{G}_2$. It is easy to show that $f(xy) \neq f(x)f(y)$ for $x = y = -1 \in \mathbb{G}_1$.

This proves that f is not a fuzzy homomorphism.

Definition 2.15: Let $\mathbb{G}$ be a fuzzy bornological group. For $g \in \mathbb{G}$ there is the right translation map $r_g: \mathbb{G} \to \mathbb{G}$ defined by $r_g(x) = xg$ and the left translation $l_g: \mathbb{G} \to \mathbb{G}$ defined by $l_g(x) = gx$.

Proposition 2.16: The following maps are fuzzy bornological isomorphisms from fuzzy group bornology $\mathbb{G}$ to itself for all $g \in \mathbb{G}$ where**,** $g \in \{x: \mathbb{G}(x) = \mathbb{G}(e)\}$.

(1) All left (right) translations in fuzzy bornological group.
(2) Also, the map $x \to x^{-1}$.
(3) And the inner automorphism map $x \to gxg^{-1}$

are fuzzy bornological isomorphisms.

Proof:

(1) The right translation is a bijection

such that, $\forall\ g \in \mathbb{G}$, $r_g: \mathbb{G} \to \mathbb{G}$ defined by $r_g(x) = xg$.

Then, $r_g(\mathbb{G})(x) = \left\{\sup \mathbb{G}\,(z): z \in {r_g}^{-1}(x)\right\} = \mathbb{G}(xg^{-1})$

$$\geq \min(\mathbb{G}(x), \mathbb{G}(g^{-1})$$

$$\geq \min(\mathbb{G}(x), \mathbb{G}(e))$$

$$= \mathbb{G}(x) = \mathbb{G}(xg^{-1}g)$$

$$\geq \min\left(\mathbb{G}(xg^{-1}), \mathbb{G}(g)\right)$$

$$= \mathbb{G}\ ((xg^{-1}) = r_g(\mathbb{G})(x).$$

Thus $r_g(\mathbb{G}) = \mathbb{G}$.

Clearly $r_{gh} = r_g \circ r_h$ and $r_{g^{-1}} = (r_g)^{-1}$.

Since, $(r_g \circ r_{g^{-1}})(x) = r_g(r_{g^{-1}}(x)) = r_g(xg^{-1}) = xg^{-1}g = r_e(x) = x = Id_G$. The inverse of r_g is also bounded, that is r_g is a fuzzy bornological isomorphism.

Similarly l_g is a fuzzy bornological isomorphism.

(2) For the inverse map.

Since $f(\mathbb{G})(y) = \{sup\mathbb{G}(z): z \in f^{-1}(y)\} = \mathbb{G}(y^{-1}) = \mathbb{G}(y)$

for all $y \in \mathbb{G}, f(\mathbb{G}) = \mathbb{G}$.

Since $f^{-1}(x) = x^{-1}$ is a fuzzy bounded by definition of a fuzzy bornological group.

The inverse of the inverse map is itself, so it has bounded inverse.

Thus the inverse map is a fuzzy bornological isomorphism.

(3) The automorphism map. This is a composition of $r_{g^{-1}}$ and l_g, thus is a fuzzy bornological isomorphism.

Any map between two fuzzy bornological groups is fuzzy bornological isomorphism.

Corollary 2.17: Let $\mathbb{G}$ be a fuzzy bornological group and $g_1, g_2 \in \mathbb{G}$. There exists a fuzzy bornological isomorphism f such that $f(g_1) = g_2$.

Proof:

r_g is a fuzzy bornological isomorphism $\forall g \in \mathbb{G}$ (by Proposition (3.16))

if $f = r_{g_1{}^{-1}g_2}$, we get $f(g_1) = g_2$.

Theorem 2.18: A fuzzy bornological group is homogeneous space.

Proof:

Let $x_1{}^{-1}x_2 = a \in \mathbb{G}$ and consider the map $f: (\mathbb{G}, \bar{\beta}) \to (\mathbb{G}, \bar{\beta})$ defined by $f(x) = xa$, then f is a fuzzy bornological isomorphism by Proposition (3.16), which implies that the space $\mathbb{G}$ is homogeneous since, $f(x_1) = x_2$.

Proposition 2.19: Let $\bar{H}$ any fuzzy bounded set of $\mathbb{G}$ and $\bar{A}$ any fuzzy set of FBG. Suppose $a \in \{x: \mathbb{G}(x) = \mathbb{G}(e)\}$. Then $a\bar{H}, \bar{H}a, \bar{A}\bar{H}, \bar{H}\bar{A}$ are fuzzy bounded sets.

Proof:

Academic Science Journal

Let $f: \mathbb{G} \to \mathbb{G}$ be a map defined by $f(x) = ax$. Since f is a fuzzy bornological isomorphism (by Proposition (3.16)), then $f(\bar{H}) = a\bar{H}$ is a fuzzy bounded.

Since $\bar{H}\bar{A} = \bigcup_{a \in \bar{A}} r_a(\bar{H})$, the conclusion follows.

Similarly we may prove the remaining in the case when $f(x) = xa$.

So for any fuzzy subset $\bar{A}$ of $\mathbb{G}$, the fuzzy set $\bar{H}\bar{A}$ and $\bar{A}\bar{H}$, $a\bar{H}, \bar{H}a$ are fuzzy bounded sets.

Theorem 2.20: Let $(\mathbb{G}, \bar{\beta})$ be a fuzzy bornological group and $\mathbb{K}$ be a fuzzy subgroup of $\mathbb{G}$. Then the relative fuzzy bornology $\bar{\beta}_{\mathbb{K}}$ is a fuzzy bornological group on $\mathbb{K}$.

Proof:

Let $\mathbb{K}$ is a constant fuzzy bounded set s.t. $\mathbb{K}(x) = 1 \ \forall x \in \mathbb{K}$.

We must show that $(\mathbb{K}, \bar{\beta}_{\mathbb{K}})$ is a fuzzy bornological group such that,

$$\bar{\beta}_{\mathbb{K}} = \{ \bar{B} \cap \mathbb{K} : \ \bar{B} \in \bar{\beta}\}.$$

Now to show $(\mathbb{K}, \bar{\beta}_{\mathbb{K}})$ is a fuzzy bornological group.

Let $\bar{B}_1, \bar{B}_2 \in \bar{\beta}$, the product map:

$$\alpha\big((\bar{B}_1 \cap \mathbb{K}) \times (\bar{B}_2 \cap \mathbb{K})\big)(z) = \sup((\bar{B}_1 \cap \mathbb{K})(x) \cap (\bar{B}_2 \cap \mathbb{K})(y)) : \alpha(x, y) = z\}$$

$$= \sup\{((\bar{B}_1 \cap \bar{B}_2) \cap \mathbb{K})(xy) : xy = z\}$$

$$= \big((\bar{B}_1 \cap \bar{B}_2) \cap \mathbb{K}\big)(z) \subseteq (\mathbb{K}, \bar{\beta}_{\mathbb{K}}).$$

Which it is a fuzzy bounded set.

The inverse map:

$$\alpha(\bar{B}_1 \cap \mathbb{K})(z) = \sup \{ (\bar{B}_1 \cap \mathbb{K})(x) : \alpha(x) = z\}$$

$$= \sup\{ (\bar{B}_1 \cap \mathbb{K})(x) : x^{-1} = z\}$$

$$= \sup\{\bar{B}_1 \cap \mathbb{K})\,(x) : x = z^{-1}\}$$

$$= (\bar{B}_1 \cap \mathbb{K})(z^{-1}) = (\bar{B}_1 \cap \mathbb{K})^{-1}(z) \subseteq (\mathbb{K}, \bar{\beta}_{\mathbb{K}}).$$

Which it is a fuzzy bounded set.

Then $(\mathbb{K}, \bar{\beta}_{\mathbb{K}})$ is a fuzzy bornological group.

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